On the Pure Theory of Wage Dispersion

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Abstract

We study an equilibrium model of the labor market with identical firms and homogeneous workers, and with search and on-the-job search. Jobs are dynamic contracts that allow firms to match the worker's outside offers or let the job be terminated. For a non-degenerate distribution of wage offers to arise in the environment, it is necessary and sufficient that (i) there be a positive cost of job turnover, in terminating an existing job, or in posting a new one; and (ii) there is limited counteroffering to the worker's outside offers. The model is calibrated to the U.S. labor market to produce a wage offer distribution that resembles observations, together with a distribution in the wages earned that is consistent with data. The model also suggests that policies that impose larger costs on hiring and termination reduce wage dispersion and the mean wage offered, whereas technologies that facilitate job matching and posting increase them.

Keywords: wage dispersion, search, on-the-job search, dynamic contracting

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1 Introduction

Can search and wage posting support a non-degenerate equilibrium distribution of wages offered in a labor market with identical firms and homogeneous workers? The search for a *pure* theory of wage dispersion starts from Diamond (1971), who offers a negative answer to the above question. The only equilibrium he finds, in an environment that meets the qualifications of the question, is one in which only the monopsony wage is offered.

As is well put by Rogerson, Shimer and Wright (2005, page 976), a pure theory of wage dispersion is of interest for two reasons. "First, the early literature suggested that search is relevant only if the distribution from which you are sampling is non-degenerate, so theorists were naturally led to study models of endogenous dispersion. Second, many people see dispersion as a fact of life, and for them the issue is empirical rather than theoretical." A pure theory of wage dispersion may provide an explanation for the "unexplained" wage differences in the labor market. Mortensen (2003, page 1) reports that "Observable worker characteristics that are supposed to account for productivity differences typically explain no more than 30 percent of the variation in compensation."

Burdett and Mortensen (1998) (hereafter BM) develop a first pure theory of wage dispersion. They show that adding on-the-job search to Diamond (1971) would produce a non-degenerate distribution of equilibrium wage offers. Their idea is that, relative to lower wages, higher wages, while imposing greater costs of labor compensation on the firm, also offer the benefits of lower employment turnover – the probability with which the worker quits from his current job decreases in the pay of that job. This trade off between compensation and job turnover results in differential wages offered in equilibrium by identical firms to identical workers.

In BM, an employment contract is a promise of constant wage until the worker finds a better outside offer and quits his current job. Thus the dispersion in wages is essentially a dispersion in the compensation contracts offered. Burdett and Coles (2003) (hereafter BC) generalize BM, allowing the contract to optimize on two dimensions: the initial wage offer, and the profile of continuation wages as a function of the worker's tenure at the firm. Their environment produces not only a non-degenerate equilibrium distribution of initial wages offered, but also continuation wages that are monotonically increasing in the worker's tenure.[1]

The theories of BM and BC, however, are subject to a few limitations. First, in

[1]Stevens (2004) considers a similar model where workers are risk neutral. There is a pure wage dispersion with a two-point support, and there is no job turnover. Like BC, Stevens (2004) assumes that firms post dynamic contracts to which they can commit, but does not allow the contracts to respond, ex post, to the worker's outside offers.

both BM and BC, the continuations of the initial contract the firm offers, whether they be a constant wage until the worker quits voluntarily, or a wage-tenure profile where wage increases in the worker's tenure, are not allowed to be dynamically contingent on the outside offers the worker receives ex post. Once the job starts, the firm never responds to the worker's outside offer.[2] Obviously, if one stipulates that the worker's outside offers are observable and the labor contract is allowed to be contingent on the worker's history with the firm, including all offers he receives on the job, past and current, then the contracts in BM and BC may not be optimal. Second, even if one takes as given the contract imposed, the steady-state objective they assume for the firm to maximize is not consistent with the contract used – as we shall demonstrate in the paper. Third, in both BM and BC, workers and firms do not discount future payoffs. Fourth, neither paper is explicit about their model's informational structure with respect to the worker's outside offers, neither were they concerned about the costs that may arise in terminating an old worker or hiring a new one.[3]

What would truly arise from the BM model? Would a non-degenerate wage distribution still emerge as an equilibrium outcome of their labor market if the contract is optimally designed, allowing in particular the firm to respond optimally to the worker's outside offers, with its objective correctly formulated, with the model's information structure explicitly specified, with or without discounting, and with or without the presence of the costs commonly believed to be relevant in labor market transactions? We seek to answer these questions. We explore theoretically what sorts of specific environments, in the general framework that the model describes, would give rise to a non-degenerate distribution of wage offers. We also ask, quantitatively, whether the non-degenerate dispersion of wage offers that does arise in the model is consistent with data, in magnitude and shape.

Three results emerge from the analysis.

Result 1: *For a non-degenerate distribution of wage offers to arise in the model, it is necessary and sufficient that (i) there are positive costs of worker turnover, in terminating an existing job, or in posting a new one; and (ii) there is limited counteroffering to the worker's outside offers, where the limitedness being either imposed on the firm by private information – in the case of discounting, or as a result of voluntary firm choice – in the case of no discounting.*

[2] One way to justify this is to argue that the worker's outside offers may not be observable by third parties and so the contract cannot be made contingent upon their values. Such an argument would not seem adequate, for even if the worker's outside offers are privately observed, a complete contract should be contingent on the worker's self-reported outside offers while providing incentives for truth-telling.

[3] Their assumptions of zero discounting and the objective function of the firm in the steady state render the effects of the costs of job posting null.

Condition (i) is needed because without it the trade-off of BM between compensation and turnover would not be there to support the desired dispersion. Condition (ii) is needed because, without it, in equilibrium any superior outside offer that the employed worker receives, which is optimal for a vacant firm to make, would also be feasible and optimal for the incumbent firm to match. This deters that outside offer from being made in the first place, rendering the equilibrium be such that vacant firms decide not to target employed workers as potential new hires, and this breaks the BM story.

Consider the environment with discounting ($\beta < 1$) and publicly observed outside offers. In this case, the optimal contract is written to be fully dependent on the worker's outside offers, and the model has a unique equilibrium where the distribution of wages offered is degenerate, and all workers are offered the monopsony wage, as in Diamond (1971). Suppose outside offers are observed by the worker who receives them, but not the firm who employs the worker. In this case, private information distorts the firm's optimal response to the worker's outside offers, as state (outside-offer) contingent counteroffers are not incentive compatible and hence not made in the model's equilibrium. As such, the optimal way to respond to outside offers is to move the risk averse worker's expected utility deterministically and monotonically up in time, as in BC. This results in a unique equilibrium of the model where vacant firms offer differential wage-tenure contracts to target both employed and unemployed workers.

With $\beta = 1$ and publicly observed outside offers, firms are indifferent between retaining and terminating a worker who has received an outside offer that dominates his current contract – with no discounting the costs that the firm incurs between terminating an ongoing contract and starting the next is negligible (zero). This indifference results in differential counteroffer policies being equally optimal for the firm, giving rise to multiple equilibria of the model, with degenerate and non-degenerate distributions of the wages/contracts offered. In these equilibria, wage dispersion arises if and only if the firm chooses not to counter all of the worker's outside offers that are superior to his current contract.

Result 2: *The model makes quantitative sense. It can be calibrated to the U.S. labor market to produce a wage offer distribution that resembles observations, together with a distribution of wages earned that is consistent with the data.*

The calibration targets observed worker flows and stocks, and matching the mean-min ratio of wages earned – which the literature has used to measure the magnitude of observed dispersion in earned compensation. With $\beta < 1$ and private outside offers, the calibrated model produces a distribution of wages offered that has a unimodal density, resembling data. It also produces a distribution of wages earned with a density that is increasing and convex, not consistent with data. With private information, wages earned, governed by a wage-tenure scheme which gives the worker a raise independent of his outside offer, grow "too fast" to send too many workers to the right end of the distribution.

With $\beta = 1$ and publicly observed outside offers, as mentioned earlier the model has multiple equilibria that are associated with differential counteroffer policies of the firm, and this offers possibilities for a *right* labor market equilibrium where there is enough counteroffering (state-contingent responses to the worker's outside offers) to hold the wages earned from rising too fast up with the worker's tenure at the firm, but there is not too much counteroffering to kill the equilibrium dispersion in the wages offered.

We focus on a subset of these equilibria, where firms promise to match outside offers up to a pre-specified threshold, leaving the worker's wage constant if a superior outside offer is not received. This threshold is meant to capture the idea that, in practice, the firm's willingness and ability in making counteroffers are constrained, for reasons that are not explicitly given in this model. Within these equilibria, unless the firm promises to match all outside offers, the distribution of the wages offered is not degenerate. It also holds that if firms respond more aggressively to their workers' outside offers, then there is in equilibrium a smaller dispersion in the wages they offer.

Indeed, when the model is calibrated to the U.S. labor market with the *right* threshold for counteroffering, (matching again the observed labor market flows and stocks, and the mean-min ratio of wages earned, as for the case of $\beta < 1$), it produces, first time in literature, equilibrium distributions of both the wages offered and wages earned that resemble the data – each with a density that is unimodal and looks truncated log-normal.

Result 3: *The model suggests that policies that impose larger costs on hiring and termination could reduce wage dispersion and the mean wage offered; whereas technologies that facilitate job matching and posting could increase wage dispersion and the mean wage offered.*

In the case of no discounting for example, with or without private information regarding the worker's outside offers, the model has an equilibrium with constant wages where larger costs of worker turnover give rise to smaller wage dispersion, reducing simultaneously the support and the mean and variance of the wages offered. This result thus sheds light on why, relative to the U.S., most European countries, with heavier regulations and less flexibility in their labor markets, have less wage dispersion but more unemployment, as discussed in for example Bertola and Ichino (1995). This result also indicates that labor market policies that increase the costs of termination, while reducing wage inequality, may affect worker welfare negatively, with lower average wages offered and higher unemployment rates.

In summary, a pure theory of wage dispersion makes both theoretical and quantitative sense. A non-degenerate distribution of wage offers is not only shown to exist theoretically, but also quantitatively consistent with observed worker flows, stocks and distributions that characterize the U.S. labor market. A key variable in this theory is the firm's optimal responses to its worker's outside offers. These responses, missing in earlier research, play a critical role in giving the theory the ability to match data, especially on the distributions of wages offered and earned.

Targeting the observed distributions of both the wages offered and earned has gone beyond the scope of the initial questions of Diamond (1971) and BM. Standard search-matching models have difficulties producing the observed wage dispersions among individual workers. Hornstein et al. (2011) show that they generate only a very small, about 3.6%, differential between the average and the lowest wages paid in the U.S. labor market. And the observed Mm ratio - the ratio between the average wage and lowest wage paid - is at least twenty times larger than what the model is able to generate. Our model, calibrated to the U.S. data, generates not only the right Mm ratio, but also the observed shape for the distribution of the wages earned.

1.1 BM and BC

Why does the BM conjecture fail in the case of $\beta < 1$ and when outside offers are publicly observable? The story is as follows. In both BM and BC, since the contract does not allow for ex post adjustments in its continuation to the worker's outside offers, higher costs of compensation necessarily imply lower probabilities/costs of termination. Once this restriction is lifted and the firm is free to match, ex post, the worker's outside offers, the optimal contract, as we show in the paper, is then able to achieve a given probability of worker retention with a lower initial wage plus a promise to match the worker's outside offers.

Although the above argument is constructed under the assumption of identical firms, the logic holds more generally. Suppose some firms can make (identical) workers more productive – but not too much more productive – than other firms. Then the same logic applies and it continues to hold that the firm's ability to counter the worker's outside offers could deter the offers from being made in the first place, rendering an equilibrium where only the monopsony wage is offered.

This logic, however, does break down if outside offers are private to the worker. With private information, the optimal contract never matches the worker's outside offers and, as a result, the negative relationship between the worker's promised utility and the probability of termination in BM is restored and a dispersion in the wages offered arises.

Under the assumption of no discounting, our analysis again produces outcomes that depart substantially from BM and BC. (i) The equilibrium with a wage-tenure contract in BC is no longer an equilibrium once the firm's objective/value function is correctly formulated. (ii) Although the fixed-wage equilibrium in BM is indeed an equilibrium in our model (i.e., derived correctly, with a positive cost of termination, under publicly or privately observed outside offers), the model also has a class of equilibria in which a fraction of the vacant firms offer the monopsony wage, as those in Diamond (1971), while the rest offering a dynamic contract that promises to make counteroffers up to a threshold, and this threshold increases in the worker's beginning-of-period expected utility. What's most important, of course, is that the fixed wage equilibrium of BM implies a wage dispersion that is at odds with observations, whereas our equilibrium with counteroffers can be calibrated to produce labor market outcomes that are consistent with data, in both the wages offered and earned. (iii) In BM and BC, the costs of job termination and job posting play no roles in determining the equilibrium wage distribution. In our model, a positive cost of terminating an existing job, or that of posting a new one, is essential for obtaining a non-degenerate distribution of wage offers. Without the costs, the trade-off between compensation and turnover, which BM prescribes for supporting the differential wages offered, would collapse to result in a unique equilibrium where only the Walrasian wage is offered and paid.

1.2 More on the literature

In modeling how firms react to employee outside offers and how that interacts with wages offered in equilibrium, our work is related to Postel-Vinay and Robin (2002a) where the incumbent and poaching firms engage in Bertrand competition by proposing a fixed wage for the worker. If all firms have the same productivity, then the equilibrium wage distribution has a two point support between the monopsony and Walrasian wages, although all initial wages are the same monopsony wage. Our work is also related to Moscarini (2005) where the poaching and incumbent firms (with whom the worker has differential productivities) engage in a first-price auction in which they each offer a lump-sum to the worker for the right to employ him. The winning firm and the worker would then engage in Nash bargaining to determine a new wage for the worker. Moscarini (2005) is not a pure theory of wage dispersion, for the equilibrium wage distribution is degenerate if firms are identical.

Between the papers discussed above and that of ours, a difference is in how the interactions between the incumbent and poaching firms are modeled. In the papers discussed, firms engage in ex post and face-to-face competition for the worker where they play strategically against each other. We, instead, take the worker's stochastic outside offers as exogenous and let the firm react to them through an ex ante optimally designed contract. Obviously, behind our modeling strategy is the traditional search/matching idea where jobs (contracts) are publicly posted and matched randomly with workers who search for them, and whoever accepts the offer gets the job, but the firm who posts the offer never interacts with the current employer of the worker it is matched with. What's novel here is that we make the contract dynamic and fully optimal.[4]

Moscarini and Postel-Vinay (2013) also study an equilibrium model of the labor market with on-the-job search where firms post and commit to dynamic employment contracts. Their objective is to understand how wage and firm size distributions move over the business cycle. For that, they impose an "equal treatment constraint" on the firm to pay equal wage to all of its employees. Such a constraint, while greatly reducing the dimensionality in their optimal contracting problem, lowers also the firm's ability in conditioning worker compensation on individual history. Relative to Moscarini and Postel-Vinay (2013), we allow firms to offer contracts that are contingent on all relevant states of the world, especially the worker's history with the firm.

But above all, what differentiates us most importantly from the above papers is that our model not only generates theoretically a non-degenerate wage offer distribution with identical firms and homogenous workers, but also proves successful in matching data, on the major flows and stocks of the labor market, and on the distributions of wages offered and earned.[5]

The model is laid out in Section 2. Section 3 studies the model under the assumption that firms and workers discount future payoffs. Section 4 considers the case where firms and workers do not discount future payoffs. Sections 5 discusses the potential relevance of the model for explaining the well known difference between the European and the U.S. labor markets in wage differentials. Section 6 concludes the paper.

[4]In the bidding games of Postel-Vinay and Robin (2002a) and Moscarini (2005), the outcome of the competition between the incumbent and poaching firms is independent of the worker's history of employment. Thus, risk sharing, which is an important part of many employment relationships, is not considered. The optimal contract in our model, in contrast, is designed to achieve the most efficient combination of incentives and risk sharing between the risk neutral firm and the risk averse worker.

[5]When the model of Postel-Vinay and Robin (2002a) is tested empirically in Postel-Vinay and Robin (2002b), the estimated discount rate is extremely high, from 30% to 55% for different occupational groups in the case of risk neutrality. From the perspective of Hornstein et al. (2011), this can be interpreted as that the value of being unemployed must be extremely low for their model to generate the observed wage dispersion, just as with the standard search model. When the model of Moscarini (2005) is calibrated to the U.S. data in Moscarini (2003), it generates a mean-min ratio of 1.16, much lower than the observed value.

2 Model

The basic structure of the model is almost exactly Burdett and Coles (2003), except we allow agents to discount future payoffs and, in addition, time is discrete in our model but continuous in theirs. The model differs from Burdett and Mortensen (1998) also in that we assume risk averse workers and their workers are risk neutral.

Let t denote time: $t = 1, 2, \cdots$ There is a single perishable consumption good in the model. The economy has a continuum, with unit mass, of identical workers who belong to an infinite sequence of overlapping generations. Each worker, when alive, has a constant probability $1 - \delta \in [0, 1]$ to survive into the next period (i.e., δ is the constant mortality rate). Workers who die are replaced immediately by an identical young worker. All workers have the following preferences:

$$\mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} (\beta(1-\delta))^{t-\tau} u(c_t) \right],$$

where $\mathbb{E}_\tau$ denotes the worker's expectation conditional on information available at the beginning of period τ, $\tau \geq 1$; $\beta \in [0, 1]$ is the discount factor; c_t and $u(c_t)$ denote, respectively, the worker's consumption and utility in period t. Assume $c_t \in \mathbb{R}_+$ for all t. That is, consumption must be non-negative.[6] Last, assume the utility function u is bounded, strictly increasing, strictly concave, twice differentiable, and satisfies the Inada conditions.

The economy also has a collection of identical firms who live forever and maximize expected profits, using the same discount factor β. Each period, each firm can employ one worker to produce a constant output of $\theta(\geq 0)$ units of the good. The measure of these firms is determined subject to free entry and exit. Existing firms are free to exit the economy at the end of any period if they wish, and profitable new firms can be created at no costs to join the economy at the beginning of any period.

A labor market opens at the beginning of each period where workers and firms are matched to form productive pairs. All workers, employed and unemployed, can participate in this market at zero costs. Vacant firms, however, must pay a fixed cost of $k(\geq 0)$ units of the consumption good each period to post a vacancy in the market. Any vacancy posted is an employment contract that the firm would offer to the potential worker it might be matched with. Whoever the firm is matched with gets offered the contract.

Matchings are random. Each period, the labor market produces $M(1, v)$ units of matches, where 1 is the measure of all workers and v is that of the vacant firms.

Upon a successful match, if the worker is currently employed, his employer has the

[6] What is important is that the worker's consumption is bounded from below, but not specifically by zero.

option to respond to the offer he receives. The worker stays with his current employer if the latter provides a counter offer that dominates his outside offer; otherwise he quits to pursue the outside offer. Note that in this process, each party gets to move once, the poaching firm first, the incumbent firm next, and the worker last.

The contract can be fully dynamic and the employment relationship may end in one of two scenarios. One, the worker dies. Two, the worker is terminated according to the terms of the contract. The latter scenario may include two cases: the worker quits voluntarily to take a better outside offer he receives on the job which the firm refuses to match; and the firm terminates the worker to send him to unemployment.

Any termination imposes a non-negative cost $C_0(\geq 0)$ on the firm. Imagine the firm who terminates a worker (at the end of period t) to go back to the labor market (at the beginning of period $t+1$) to post a vacancy. Then the total cost he pays is $C_0 + \beta k$.

We make the following assumptions on contracting.

Assumption 1. ***(Limited Liability)*** *The worker's compensation is non-negative.*

Assumption 2. ***(Limited Commitment)*** *In each period, the worker is free to walk away from the contract, before and after receiving his outside offer. The firm, on the other hand, is fully committed to the terms of any contract it offers.*

Assumption 2 implies that, if the firm wants to retain a worker who has an outside value of ξ, then the continuation of the contract must promise the worker expected utility of at least ξ. Note that Assumptions 1 and 2 are both imposed, explicitly or implicitly, in Burdett (1978), Burdett and Mortensen (1998), and Burdett and Coles (2003).

3 Discounting ($\beta < 1$)

In this section, the model is studied under the assumption of $\beta < 1$. We look first at the case where the worker's outside offers are publicly observed, then the case where they are privately observed, and then we try calibrating the model to the U.S. data.

3.1 Public outside offers

Assumption 3. ***(Public Outside Offers)*** *Any outside offer is public information between the worker who receives it and the firm who employs him.*

3.1.1 The Labor Market

To start the analysis, we describe the aggregate variables of the labor market whose values will be taken as given, in the stationary equilibrium we are about to define, by firms and workers in their individual decision making.

Let $u \in [0,1]$ denote the measure of unemployed workers in equilibrium and $1-u \in [0,1]$ that of employed workers. Let

$$p_w = M(1, m-(1-u)) \in [0,1] \tag{1}$$

denote the probability with which a worker, employed or unemployed, is matched with a (vacant) firm in equilibrium. Remember once a firm is matched with a worker, employed or unemployed, the firm automatically offers him the contract that has been posted. Let $p_f \in [0,1]$ denote the probability with which an individual vacant firm is matched with a worker, employed or unemployed, in equilibrium. That is,

$$p_f = \frac{M(1, m-(1-u))}{m-(1-u)} \in [0,1]. \tag{2}$$

At the beginning of each period, in the labor market there is a distribution of vacant firms in the starting expected utility they post for new hires. The support of this distribution is denoted Φ^*, which is the set of expected utilities that vacant firms are able to deliver and offer in equilibrium. For each $\xi \in \Phi^*$, let $F^*(\xi)$ denote the fraction of vacant firms that post a job that offers the worker an expected utility no greater than ξ. Assume F^* has a density denoted $f^* : \Phi^* \to \mathbb{R}_+$.

Note that all ξs in Φ^* may not be offered in equilibrium with positive probability, but we require that all offers be feasible for the vacant firm to deliver. In other words, for any $\xi \in \Phi^*$, there exists a feasible contract, the notion of which to be given in the next subsection, that gives the worker expected utility ξ. Note also that we allow firms to use symmetric but mixed strategies for job posting. As such, each expected utility posted, and subsequently offered, is simply a random draw from the distribution F^*, with F^* being the equilibrium mixed strategy used for job posting by all vacant firms.

At the beginning of any period, there is also a distribution of employed workers in the expected utility their employer has promised to deliver. Let $G(V)$ denote the equilibrium fraction of employed workers who are promised by their current employer an expected utility no greater than V, for all $V \in \Phi^*$.

Finally, let V_0 denote the expected utility for an unemployed worker at the beginning of a period in equilibrium. We have $V_0 \in [V_{\min}, V_{\max})$ and

$$V_0 = u(0) + \beta(1-\delta)\left[p_w \int_{\Phi^*} \max\{\xi, V_0\} dF^*(\xi) + (1-p_w)V_0\right]. \tag{3}$$

Remember the unemployed worker's consumption is normalized to zero. With probability p_w the unemployed worker is matched with a vacant firm to receive a random offer of expected utility $\xi \in \Phi^*$. He would take this offer if the value of this offer is above his reservation utility, which is V_0, and reject it to remain unemployed otherwise. With probability $1-p_w$ he is not matched with a vacant firm and he then remains unemployed moving into the next period.

3.1.2 Equilibrium Contracting

In the labor market, each vacant firm posts an expected utility, together with a contract which delivers that expected utility, that it promises to offer to any worker, employed or unemployed, whom it expects to be randomly matched with. In choosing the starting expected utility and the corresponding contract, the firm takes as given, in addition to the parameters of the physical environment, the aggregate states of the labor market, including in particular the contract used in equilibrium which we denote as σ^*, and the distribution $F^* : \Phi^* \to [0,1]$ of the starting expected utilities posted/offered by individual firms in equilibrium (through the equilibrium contract σ^*).

In formulating the individual vacant firm's contracting problem, we take the stand that the contract must be able to respond in each period to any outside offer that the firm perceives to be feasible for other vacant firms to offer and hence its worker to receive in equilibrium. That is, each individual contract must and need only consider the ξs in Φ^*, the set of expected utilities that are feasible for the equilibrium contract to deliver. With this, in equilibrium, for any individual firm, an employment contract, formulated recursively, following the tradition of Green (1987) and Spear and Srivastava (1987), is as follows:

$$\{c(V), I(\xi;V), V_r(\xi;V), I(V), V_n(V) : \xi \in \Phi^*, V \in \Phi\}, \tag{4}$$

where (i) V, the "state variable", denotes the expected utility of the worker that the continuation of the contract promises to deliver at the beginning of the period, and the set $\Phi \subseteq \Sigma \equiv \left[\frac{u(0)}{1-\beta(1-\delta)}, \frac{u(\infty)}{1-\beta(1-\delta)}\right)$ denotes the set of all Vs that are feasible for *this* contract to deliver - the state space of this contract. Note that at this stage of individual contracting, Φ is not Φ^*, although later the Φ for the optimal contract will be required to be consistent with Φ^* in equilibrium. At this stage, Φ is an endogenous part of the individual contract which takes Φ^*, the set of outside offers that it perceives for his worker to receive in equilibrium, as given.

(ii) ξ denotes the worker's current outside offer. Each ξ is drawn from the set Φ^*, the set of all possible outside offers in equilibrium.

(iii) $c(V)$ denotes the worker's compensation in the period.

(iv) $I(\xi;V) \in \{0,1\}$ indicates the worker's status with the firm after receiving outside offer ξ. If $I(\xi;V) = 1$, the worker is retained. If $I(\xi;V) = 0$, he is terminated.

(v) $V_r(\xi;V)$ denotes the worker's next period promised utility if he is retained, i.e., if $I(\xi;V) = 1$. Note if the worker is terminated upon ξ, then his next period expected utility is simply $\max\{\xi, V_0\}$.

(vi) $I(V) \in \{0,1\}$ indicates the worker's status with the firm after receiving no outside offer. Specifically, if $I(V) = 1$, the worker is retained and given expected utility $V_n(V)$. If $I(V) = 0$, the worker is terminated and his value is V_0. The value of the worker conditional on not being matched with a firm is $I(V)V_n(V) + (1 - I(V))V_0$.[7]

We now formulate optimality. For each $V \in \Phi$, let $U(V)$ denote the maximum (normalized) expected value of the firm given that the worker it currently employs is promised expected utility V. Next, for all $\xi \in \Phi$, let $\overline{U}(\xi)$ be the (normalized) expected value for the vacant firm who posts a contract that offers expected utility ξ. We have, as is straightforward to calculate, that for all $\xi \in \Phi$,

$$\overline{U}(\xi) = \frac{-(1-\beta)k + p_f \gamma(\xi) U(\xi)}{1 - (1 - p_f \gamma(\xi))\beta}, \tag{5}$$

where remember p_f is the probability with which vacant firms are matched with a worker (employed or unemployed), and $\gamma(\xi) \in [0, 1]$ is the probability with which a contract that offers expected utility ξ is accepted, upon being offered to a randomly matched worker.

We call $\gamma(\xi)$ the acceptance probability for the offer ξ. Note, importantly, that the firm takes both p_f and $\gamma(\cdot)$ as given. Note also that since the set Φ is a choice variable for the firm, we take as given that the domain of the function $\gamma(\cdot)$ is Σ.

The value function $U : \Phi \to \mathbb{R}$ and the optimal contract must then solve the following Bellman equation: $U = \Gamma U$, where for all $V \in \Phi$,

$$\begin{aligned} \Gamma U(V) \quad = \quad & \max_{c, I(\cdot), V_r(\cdot), I, V_n} (1-\beta)(\theta - c) + \beta\delta\left[\pi - (1-\beta)C_0\right] \\ & + \beta(1-\delta)p_w \int_{\Phi^*} I(\xi)U(V_r(\xi)) + (1 - I(\xi))\left[\beta\pi - (1-\beta)C_0\right] dF^*(\xi) \\ & + \beta(1-\delta)(1-p_w)\left\{IU(V_n) + (1-I)\left[\beta\pi - (1-\beta)C_0\right]\right\} \end{aligned} \tag{6}$$

subject to

$$\begin{aligned} u(c) + \beta(1-\delta)p_w \int_{\Phi^*} & I(\xi)V_r(\xi) + (1 - I(\xi))\max\{\xi, V_0\} dF^*(\xi) \\ & + \beta(1-\delta)(1-p_w)[IV_n + (1-I)V_0] = V, \end{aligned} \tag{7}$$

$$c \geq 0, \tag{8}$$

$$I(\xi)(1 - I(\xi)) = 0, \; \forall \xi, \tag{9}$$

$$V_r(\xi) \in \Phi, \; \forall \xi \text{ with } I(\xi) = 1, \tag{10}$$

[7] Note that instead of treating the state of no-outside-offer as a separate state, alternatively we could treat it as a state in which the worker receives an outside offer of a very low value, say $\xi = 0$. It appears however that treating the case of no-outside-offer as a separate state is more convenient for formulating the outcomes of random matching.

$$V_r(\xi) \geq \max\{\xi, V_0\}, \forall \xi \text{ with } I(\xi) = 1, \tag{11}$$

$$I(1 - I) = 0, \tag{12}$$

$$V_n \in \Phi, \tag{13}$$

$$V_n \geq V_0, \tag{14}$$

where Φ is the largest self-generating set with respect to the constraints (7)-(14) and $\Phi \subseteq \Sigma$,[8] and

$$\pi \equiv \max_{\xi \in \Phi} \overline{U}(\xi), \tag{15}$$

where the values $\overline{U}(\xi)$ are given in (5). In (6), π is the value of being vacant for the firm. Being vacant, the firm picks an optimal ξ from the set of deliverable values Φ to post, as equation (15) describes. Notice that in any case of termination, the firm incurs immediately the cost C_0 and then moves into the next period to get the value π. Equation (7) is a promise-keeping constraint that requires the choices of the current variables be consistent with the definition of V. Notice that in the case of termination, the worker is free to take the outside offer or to become unemployed. So any outside offer below V_0 would never be taken, and thus should never be offered. Put differently, the probability that the worker receives an outside offer below V_0 is zero. Equation (10) says that what the firm promises to the worker must be what the contract could deliver. Equation (11) is a self-enforcing constraint: the contract must give the worker better than his outside offer in order to retain him.[9]

A particularly important variable in the individual firm's problem of optimal contracting is $\gamma(\xi)$, the probability that a contract offering expected utility ξ gets accepted upon a successful match. Notice that $\gamma(\xi)$ enters equation (5) in determining the value of the firm who offers ξ. Note that the values of $\gamma(\xi)$, as those of the other equilibrium objects, including σ^*, F^*, V_0, p_w, p_f, and u, are taken as given by individual firms in their optimal decision making. Question is, what values would individual firms assign to $\gamma(\xi)$, for all $\xi \in \Phi$ and all $\Phi \subseteq \Sigma$?

Observe first that $\gamma(\xi) = 0$ for all $\xi < V_0$. In words, no worker, employed or unemployed, would accept a contract with expected utility ξ strictly less than his reservation utility V_0.

[8]See Abreu, Pearce and Stacchetti (1990) and Wang (1995).

[9]The formulation of the optimal contract builds on many earlier works in the literature, including Thomas and Worrall (1988), Phelan (1995), Kocherlakota (1996), Ray (2002), and Clementi and Hopenhayn (2006) on dynamic contracting with limited commitment, and Spear and Wang (2005), DeMarzo and Fishman (2007), Wang (2011), and Wang and Yang (2012, 2015a) on dynamic contracting with optimal termination.

Suppose then $\xi \geq V_0$. An unemployed worker would always accept such an offer. Whether an employed worker would accept the offer depends upon how the worker's current employer would respond to that offer. On this, there are two cases.

Case I Suppose $\xi \in \Phi^*$. This is an *on the equilibrium path* offer to which the firm has prepared to react with the contract it posts. Specifically, it would retain the worker if $I(\xi) = 1$ and terminate him if $I(\xi) = 0$. Thus,

$$\gamma(\xi) = u + (1-u) \int_{\Phi^*} [1 - I(\xi; V)] dG(V).$$

Case II Suppose $\xi \in \Phi \setminus \Phi^*$. That is, the offer the worker receives is something his current employer did not anticipate to occur (outside the set Φ^*). How then would the firm react to this ξ, an off equilibrium path offer that came as a surprise? Here we take the stand that such an offer would be viewed as a zero probability incidence and thus the worker and his current employer, upon observing it, would not change their beliefs about the distribution from which any future outside offer would be drawn (that is, they believe that any future outside offer would still be drawn randomly from the rationally perceived equilibrium distribution $F^* : \Phi^* \to [0, 1]$). As such, upon the draw of the ξ, if the worker and the incumbent firm would enter into a continuation of their initial contract which promises the worker a new expected utility of $V \in \Phi$, then the value for the firm would just be $U(V)$ - the value function $U(\cdot)$ remains valid for calculating values for the firm. Given these, if

$$\max_{V \in \Phi \text{ and } V \geq \xi} U(V) \geq \beta\pi - (1-\beta)C_0, \tag{16}$$

then the incumbent firm would retain the worker with a counteroffer of expected utility $V \geq \xi$. As such, ξ is accepted only if it is offered to an unemployed worker. Therefore $\gamma(\xi) = u$. Otherwise, the incumbent firm would just terminate the worker and so $\gamma(\xi) = 1$. Note the left hand side of (16) is the maximum value it could obtain if it retains the worker, the right hand side the value from terminating the worker.

To summarize, for any $\xi \in \Phi$,

$$\gamma(\xi) = \begin{cases} 0, \quad \text{if } \xi < V_0 \\ u + (1-u) \int_{\Phi^*} 1 - I(\xi; V) dG(V), \quad \text{if } \xi \geq V_0 \text{ and } \xi \in \Phi^* \\ u, \quad \text{if } \xi \geq V_0,\ \xi \in \Phi \setminus \Phi^*, \text{ and (16) holds} \\ 1, \quad \text{if } \xi \geq V_0,\ \xi \in \Phi \setminus \Phi^*, \text{ and (16) does not hold} \end{cases} . \tag{17}$$

3.1.3 Equilibrium Definition

We now define a stationary rational expectations equilibrium of the model, requiring the aggregate stocks and distributions, the contracts used, and the starting expected utilities offered by firms be time invariant, and that individual decisions be consistent with aggregate outcomes. Formally,

Definition 1. *A stationary rational expectations equilibrium of the economy consists of*

(i) An equilibrium contract $\sigma^* = \{c^*(V), I^*(\xi; V), V_r^*(\xi; V), I^*(V), V_n^*(V) : \xi \in \Phi^*$ *and* $V \in \Phi^*\}$ *for firms to use in equilibrium;*

(ii) An equilibrium (mixed) strategy $F^* : \Phi^* \to [0, 1]$ *of vacant firms for posting a starting expected utility for new hires;*

(iii) An equilibrium distribution of expected utilities for employed workers (those currently employed under σ^**)* $G : \Phi^* \to [0, 1]$*;*

(iv) An expected utility of unemployed workers $V_0 \in \Sigma$*;*

(v) An unemployment rate $u \in [0, 1]$*;*

(vi) An expected value for vacant firms $\pi = 0$*;*

(vii) Matching probabilities p_w *and* p_f *for workers and vacant firms respectively, and an offer acceptance function* $\gamma : \Sigma \to [0, 1]$

such that

(a) The equilibrium contract σ^* *is optimal for each individual firm (i.e.,* σ^* *solves problem (6)-(15));*

(b) F^* *is the vacant firm's optimal strategy for expected utility posting: it solves*

$$\max_{F:\Phi^* \to [0,1]} \int_{\Phi^*} \overline{U}(\xi) dF(\xi), \tag{18}$$

taking as given the optimal contract σ^* *(including* Φ^**), and the equilibrium* G*,* V_0*,* $\gamma(\cdot)$*, and* p_f*, with* $\overline{U}(\cdot)$ *given by (5);*

(c) Unemployed workers accept an offer $\xi \in \Phi^*$ *if and only if* $\xi \geq V_0$*,* V_0 *given by (2);*

(d) $p_w = M(1, m - (1 - u))$*;* p_f *satisfies (2); and* $\gamma(\cdot)$ *satisfies (17);*

(e) The distribution G *of the employed workers' expected utilities are consistent with* F^**, the equilibrium distribution of starting expected utility offers, the dynamics the equilibrium contract* σ^* *generates, and the constant mortality rate* δ*;*

(f) The equilibrium unemployment rate u *is consistent with* σ^* *and* F^* *(particularly the termination policies they dictate), as well the total measure of firms in the economy* m *and the matching function* M*.*

3.1.4 Results

The optimal contract is fully characterized, in the following proposition.

Proposition 1. *The value function $U(\cdot)$ is strictly decreasing and concave, and the following holds with the optimal contract: For all $V \in \Phi = [V_0, V_{\max})$,*

(i) There exists $\overline{\xi}(V)$ such that for all $\xi \in \Phi^*$,

$$I(\xi; V) = \begin{cases} 1, & \text{if } \xi < \overline{\xi}(V) \\ 0, & \text{if } \xi > \overline{\xi}(V) \end{cases};$$

(ii) For all $\xi \in \Phi^*$, $V_r(\xi; V) = \max\{\xi, V\}$;

(iii) $I_n(V) = 1$ and $V_n(V) = V$.

Proposition 1 says that, conditional on receiving an outside offer, the worker is retained if and only if the outside offer is below a cutoff value, $\overline{\xi}(V)$, which depends on his current expected utility. Retained or terminated, the worker's next period expected utility is the maximum between V – that he starts the current period with, and ξ – that he is offered externally. Last, the worker is retained and stays constant in expected utility if the fails to receive any outside offer.

Part (ii) of the proposition implies that, conditional on retention, the worker's expected utility increases (weakly) monotonically in time, and so does his compensation. This aspect of the optimal contract compares to the wage-tenure contract in Burdett and Coles (2003). The difference, however, is that their wage-tenure profile is deterministically fixed before the contract starts, whereas in our model, the wage-tenure profile that the optimal contract generates evolves stochastically, as a function of the worker's history of outside offers. Note, again, that Burdett and Coles (2003) do not allow firms to respond, ex post, to the worker's outside offers.

Another feature of the optimal contract, which is not explicitly stated in the above proposition, is that the cutoff $\overline{\xi}(V)$ is monotonically increasing in V, so that termination occurs on higher and higher outside offers (i.e., termination occurs with lower and lower probabilities) as V increases, or as the worker stays longer on the job.[10]

Proposition 2. *The economy has a unique stationary equilibrium where all firms post the same contract which offers expected utility $V_{\min}$, and in equilibrium all employed workers are paid the same monopsony wage $c = 0$.*

That is, under Assumption 3, the economy does not have a stationary equilibrium with a non-degenerate distribution of contract offers. All workers are offered their reservation utility and, once employed, no worker would quit his current job, as in

[10] See the proof of the proposition in the Appendix for this.

Diamond (1971). This is in contrast with the results of Burdett and Mortensen (1998) and Burdett and Coles (2003).

While the proof is in the appendix, the economic intuition behind the result is as follows. Suppose the economy does have a stationary equilibrium with a non-degenerate distribution of differential contract offers. Remember these contracts are offered indiscriminately to the employed and unemployed workers whom the firms are randomly matched with. Consider first the problem of an incumbent firm whose worker has received an offer (an employment contract) from a vacant firm. Now observe, importantly, that the vacant firm (who extends the outside offer to the worker) and the incumbent firm (who must now respond optimally to the vacant firm's offer) face exactly the same optimization problem, except that the incumbent firm, if it loses the worker, must incur an extra cost $C_0 \geq 0$ for terminating an existing employment relationship. Thus if the contract being offered to the worker is optimal for the vacant firm - which is, by assumption - it must also be optimal for the incumbent firm to use as a counter offer to retain the worker. Thus in equilibrium no outside offers will be accepted by employed workers. Any outside offer that promises a higher expected utility for the worker would be matched by the worker's current employer. And of course any offer that promises a lower expected utility will be disregarded by the incumbent firm and again the worker stays with his existing job. To summarize, in equilibrium only unemployed workers would accept any offer any vacant firm posts. Note that unemployed workers are identical and have the same reservation utility.

Consider then the problem of the vacant firm who is deciding what contract to post/offer before the market opens. Given the logic in the above paragraph, what the vacant firm should offer, (which, remember, would be accepted by unemployed workers only,) would be the contract that maximizes the firm's value subject to giving the unemployed worker an expected utility which is weakly better than his reservation utility. In fact, the starting expected utility the vacant firm offers will just be equal to the unemployed worker's reservation utility, given that the firm's maximum value is attained at and only at the unemployed worker's reservation utility. The rest of the proposition then follows immediately.

In this environment, as in Burdett and Mortensen (1998), an offer of higher expected utility potentially has three effects on the value of the firm. (a) Higher compensation costs to the firm. (b) A higher probability with which the contract is accepted by a matched worker, employed or unemployed. (c) A lower probability with which the employed worker quits (the effect of on-the-job search). In our model, an offer of higher expected utility does not imply a higher equilibrium probability of job acceptance, because in equilibrium jobs are never accepted by employed workers (reason given in the above paragraph) and unemployed workers have the same reservation utility. Question then is, would (a) and (c) exist in our model to generate the trade-off that is necessary for the equilibrium dispersion, as they do in Burdett and Mortensen (1998)?

As discussed earlier, in Burdett and Mortensen (1998), since the contract does not allow ex post adjustment in its continuation to the worker's outside offers, a smaller probability of termination necessarily implies higher costs of compensation. Now the inability of the contract to respond optimally to the worker's outside offers is important. Once this restriction is lifted, this link between the costs of compensation and the probability of termination no long exists. That is, once the contract is allowed to respond optimally to the worker's outside offers, a lower expected utility promised to the worker need not imply a higher probability of termination - the contract always has the ability to match the worker's outside offer in order to enforce continuation. In other words, a lower probability of termination need not be enforced by a higher expected utility promised to the worker. A contract that responds more aggressively to the worker's outside offers may attain a lower probability of termination with a relatively low expected utility promised to the worker.

3.1.5 An Extension of the Argument

The insight from the analysis so far is that the threat to match an outside offer that exceeds the worker's current promised utility lowers the value of such an offer. This, in turn, renders such an offer not being offered in the first place and destroys the dispersion of expected utilities received by new hires in equilibrium.

The argument was presented with the assumption of homogeneous firms. The essence of the argument, however, does not hinge essentially upon that assumption. In this section, we take a step up to show that even with firms that differ in productivity – they make workers more or less productive – counteroffers in the dynamic contract can destroy the equilibrium dispersion in worker compensation and restore the monopsony wage. In other words, firms with differential productivities may offer identical wages in equilibrium.

The argument goes as follows. Suppose, for the sake of simplicity, that there are two firm types, one being more productive than the other; but workers, again, are all identical. The more productive firms are able, and may be willing, to offer higher expected utilities to their new hires. Now from the logic of the above analysis, any expected utility a more-productive vacant firm is willing to offer is an outside offer that any more-productive incumbent firm is willing to counter. In other words, in equilibrium any job offer from a more-productive firm would not be taken by a worker employed at another more-productive firm. Now, would the more-productive vacant firm be offering anything that a worker employed at a less-productive firm would take?

It depends on whether the less productive firm is willing to counter which, in turn, depends on whether the offer is sufficiently high – higher than the threshold above which the less-productive firm is not willing to counter. And, of course, the more-productive firm is willing to make such an offer if the threshold is sufficiently low, or the difference in productivity between the two productivity types is sufficiently large.

So consider a modification of the model. Assume the worker's period output with the more-productive firm is θ_h, and with the less-productive firm θ_l, and $\theta_l < \theta_h$. Suppose the fraction of firms with the low productivity θ_l is $q \in (0,1)$, and with the high productivity, θ_h, $1-q$. We also assume $k = C_0 = 0$. When it is costly to terminate an old worker ($C_0 > 0$) or to recruit a new worker ($k > 0$), the firm would have more incentives to match an existing worker's outside offer in order to retain him. This would strengthen, instead of weakening, the case.

Proposition 3. *Suppose $k = C_0 = 0$. The model has a stationary equilibrium in which all vacant firms post a contract offering expected utility $V_{\min}$ if and only if*

$$\frac{\theta_h}{\theta_l} \leq \frac{u + (1-u)q}{(1-u)q}, \tag{19}$$

where the unemployment rate u is given by

$$(1-\delta)[(1-u) + uM(1, m - (1-u))] = 1 - u.$$

When (19) holds, that is, if firms differ but not by much in productivity, then in equilibrium all new hires are offered the same expected utility $V_{\min}$ and are paid the same monopsony wage after employment starts. Note, however, that the contracts the firms offer in equilibrium do differ, in that the more-productive firm would specify to counter higher outside offers – in case they arise which they don't in equilibrium – than the less-productive firm. In other words, although differential contracts are offered, they all offer, in equilibrium, the same monopsony wage.

3.2 Private Outside Offers

Assumption 4. ***(Private Outside Offers)*** *All outside offers any employed worker receives are his private information, not observable to the firm that employs him.*

Under Assumption 4, if the firm wishes to make the terms of the contract contingent on the outside offers the worker receives, it must induce the worker to report his outside offers truthfully.[11] This would change the structure of the optimal contract and the distribution of the expected utilities offered in equilibrium.

[11]The revelation principle, which holds in this case, allows us to focus, without loss of generality, direct employment mechanisms that implements truth-telling.

The labor market in equilibrium can be described similarly as for the case of publicly observed outside offers. We start with the vacant firm's problem of what expected utility and contract to post in the labor market, taking as given that it operates in an equilibrium of the model where there is a non-degenerate distribution of expected utilities offered. Specifically, assume the expected utilities offered in equilibrium are such that any random match generates an offer $\xi \in \Phi^*$, where Φ^*, as in the case of publicly observed outside offers, is the set of all expected utilities that a vacant firm is able to offer and deliver. The set Φ^* and the distribution of the expected utilities offered, $F^* : \Phi^* \to [0, 1]$, are known (rationally perceived) to all workers and firms in the economy.

Under Assumption 4, the terms of the contract cannot be made directly contingent on the worker's outside offers, but could instead be contingent on the reports of the worker's outside offers. Using again the worker's beginning-of-period expected utility, denoted V, as a state variable, a dynamic contract, defined recursively, is

$$\sigma = \{c(V), I(\xi; V), V_r(\xi; V), I(V), V_n(V) : \xi \in \Phi^* \text{ and } V \in \Phi\}.$$

The variables of the contract are defined similarly as those in the case of public outside offers, except that the ξ is now the report of the worker's current (privately observed) outside offer.

A contract σ is feasible and incentive compatible if, for all $V \in \Phi$,

$$u(c(V)) + \beta(1-\delta)p_w \int_{\Phi^*} I(\xi; V)V_r(\xi; V) + (1 - I(\xi; V)) \max\{\xi, V_0\} dF^*(\xi)$$
$$+\beta(1-\delta)(1-p_w)[I(V)V_n(V) + (1 - I(V))V_0] = V, \tag{20}$$

$$V_r(\xi; V) \geq \begin{cases} V_r(\xi'; V), \ \forall \xi' \text{ with } I(\xi'; V) = 1 \\ \max\{\xi, V_0\} \\ I(V)V_n(V) + (1 - I(V)) \max\{\xi, V_0\} \end{cases}, \forall \xi \text{ with } I(\xi; V) = 1, \tag{21}$$

$$\max\{\xi, V_0\} \geq \begin{cases} V_r(\xi'; V), \ \forall \xi' \text{ with } I(\xi'; V) = 1 \\ \max\{\xi, V_0\} \\ I(V)V_n(V) + (1 - I(V)) \max\{\xi, V_0\} \end{cases}, \forall \xi \text{ with } I(\xi; V) = 0, \tag{22}$$

$$I(V)V_n(V) + (1 - I(V))V_0 \geq \begin{cases} V_r(\xi'; V), \ \forall \xi' \text{ with } I(\xi'; V) = 1 \\ V_0 \end{cases}, \tag{23}$$

$$c(V) \geq 0, \tag{24}$$

$$I(\xi; V)(1 - I(\xi; V)) = 0, \; \forall \xi, \tag{25}$$

$$V_r(\xi; V) \in \Phi, \; \forall \xi \text{ with } I(\xi; V) = 1, \tag{26}$$

$$V_r(\xi; V) \geq \max\{\xi, V_0\}, \; \forall \xi \text{ with } I(\xi; V) = 1, \tag{27}$$

$$I(V)(1 - I(V)) = 0, \; V_n(V) \in \Phi, \; V_n(V) \geq V_0, \tag{28}$$

where Φ is the largest self-generating set with respect to the constraints (20)-(28).

In the above, equation (20) is the promise-keeping constraint. Equations (21)-(23) are the incentive constraints which require that the worker report truthfully whether he receives an outside offer and what the outside offer he receives is. Specifically, upon receiving any $\xi \in \Phi^*$, the worker could report an outside offer in the retention region, ξ' with $I(\xi'; V) = 1$, or in the termination region, ξ' with $I(\xi'; V) = 0$, or he could report not receiving any outside offer. In the second case, upon termination of his current job the worker could choose to accept the outside offer ξ or to become unemployed to obtain expected utility V_0.[12]

Equations (24) and (25) require, respectively, that the compensation to the worker and the termination policy be feasible. Equation (26) requires that if the worker is retained, then he must be promised an expected utility that is feasible for the contract to deliver. Equation (27) is the self-enforcing constraint which says that if the firm retains the worker, then the worker must be given no less than his outside options, ξ and V_0. Finally, equation (28) requires that the policies in the state of no outside offer be feasible and self-enforcing.

Requiring truth-telling has an immediate implication for termination. Observe that if termination occurs at any current outside offer ξ, then it must occur at any current outside offer $\xi' > \xi$. This is easy to see. Suppose not. Then, upon receiving ξ, the worker would report ξ' and will receive an expected utility weakly higher than ξ' (because of the self-enforcing constraint) and strictly higher than ξ. This breaks incentive compatibility. Similarly, if retention occurs at any ξ, then it must also occur

[12] Note that in formulating the incentive constraint, we assume that after reporting ξ' with $I(\xi'; V) = 1$ and the firm offers to retain him with expected utility $V_r(\xi'; V)$, the worker cannot quit his current job to pursue the ξ he received any more. Alternatively, we could assume that the worker can still quit after he reports ξ' with $I(\xi'; V) = 1$ and the firm offers $V_r(\xi'; V)$ to retain him. This would not change the worker's incentives to report truthfully. In fact, the worker with outside offer ξ has incentives to quit even after he reports ξ' with $I(\xi'; V) = 1$ and the firm offers $V_r(\xi'; V)$ to retain him only if $V_r(\xi'; V) < \max\{\xi, V_0\}$, which implies that the worker never has incentives to report ξ' in the first place.

at any $\xi' < \xi$. For otherwise the worker who receives ξ' would report ξ to obtain a higher expected utility. These suggest that incentive compatibility requires that workers who receive higher outside offers (above a cutoff) be terminated and lower outside offers (below the cutoff) be retained.

Truth-telling also imposes a constraint on the worker's compensation across the states in which he is retained. First, truth-telling implies that the worker's expected utility must be constant across the states of his outside offers in which he is retained. In other words, incentive compatibility rules out the possibility of making the retained worker's expected utility contingent on the state of his outside offer. Next, given the lack of commitment from the worker, the constant expected utility in the states of retention must not be lower than the cutoff for retention/termination. And finally, in order to induce truth-telling between the states of retention and termination, the constant expected utility the retained worker receives must just be equal to the cutoff for retention/termination. And these, of course, give the incentive compatible contract the very features which are essential for supporting a non-degenerate wage dispersion in Burdett and Mortensen (1998) and Burdett and Coles (2003).

The optimality of the contract and a stationary rational expectations equilibrium of the model can be formulated in a way that is parallel to that for the case of publicly observed outside offers. We leave that for the reader. We now present our main results in this section.

Proposition 4. $\Phi = [V_0, V_{\max})$. *The following holds for the optimal contract: For all* $V \in \Phi^*$,

(i) For all $\xi \in \Phi^*$,

$$I(\xi; V) = \begin{cases} 1, & \text{if } \xi < V_n(V) \\ 0, & \text{if } \xi > V_n(V) \end{cases};$$

(ii) For all ξ *with* $I(\xi; V) = 1$, $V_r(\xi; V) = V_n(V)$;

(iii) $c(V_n(V)) \geq c(V)$ *and* $V_n(V) \geq V$. *In addition, if* $f^*(V) > 0$, *then* $V_n(V) > V$.

By Proposition 4 then, the contract offered in equilibrium has the same essential features of the wage-tenure contract in Burdett and Coles (2003): it evolves along an ex ante efficiently designed dynamic but deterministic path for the cutoff for termination. Specifically, each period, conditional on the worker's current expected utility $V \in \Phi^*$, the firm sets a bar ($V_n(V)$, which is strictly above the worker's current expected utility V) for the worker's outside offer, below which the worker's next period expected utility is at that bar and above which the worker quits to pursue his outside offer. With this contract, the firm never responds to the worker's outside offers, even though it is feasible for it to do so.[13]

[13] Note that the wage-tenure contract was derived in Burdett and Coles (2003) under the assumption

Let $\underline{V}$ and $\overline{V}$ denote the infimum and supremum of the support of F^*, respectively.

Proposition 5. *Suppose $k > 0$. Then (i) the equilibrium distribution F^* does not have a mass point at $\overline{V}$; (ii) $f^*(\underline{V}) = f^*(\overline{V}) = 0$.*

As derived in the Proof of Proposition 5, the optimality of the equilibrium contract is given in the following first order condition for the employed worker's next period expected utility $V_n(V)$:

$$\begin{aligned} & p_w f^*(V_n(V))\{U(V_n(V)) - [\beta\pi - (1-\beta)C_0]\} \\ = \; & (1-\beta)[(1-p_w) + p_w F^*(V_n(V))]\left(\frac{1}{u'(c(V_n(V)))} - \frac{1}{u'(c(V))}\right), \; \forall V \in \Phi^*. \end{aligned}$$

Here, $U(V_n(V)) - [\beta\pi - (1-\beta)C_0]$ measures the firm's gains from retaining the worker with a continuation contract offering expected utility $V_n(V)$, instead of terminating him. The gains are strictly positive given that the firm would have to go through a costly process after termination to find a new worker identical to the departing worker. Notice next that the absolute value of the term $1/u'(c(V_n(V))) - 1/u'(c(V))$ measures the firm's costs, in units of current period consumption, of deviating from perfectly smoothing the worker's consumptions across the current and the next periods, conditional on retaining the worker. Notice that this term is zero if and only $V_n(Vp) = V$ so the worker's consumption is constant between the current and the next period. Thus this first order condition simply equates the expected gains and costs associated with an increase in $V_n(V)$ at the margin.

Part (i) of the proposition follows the ideas of Burdett and Mortensen (1998), except we take the stand that the worker would reject any outside offer that promises the same value as that of the ongoing contract. Otherwise, suppose the worker would accept any outside offer that promises the same value as that of the ongoing contract, as in Burdett and Mortensen (1998) and Burdett and Coles (2003). Then it can be shown that if $k > 0$ and $C_0 > 0$, the equilibrium distribution F^* does not have a mass point at $\overline{V}$. To combine the findings, as long as the cost of job-posting is positive ($k > 0$), and termination is costly (either explicitly on the firm with $C_0 > 0$, or implicitly on the worker so that he is not willing to switch between jobs that offer the same expected value), the equilibrium distribution of contract offers is non-degenerate.

Part (ii) of the proposition then says that the non-degenerate distribution has a shape which, on one dimension, is like what the economist would anticipate. We sketch the proof of this result. **Step 1.** Given (i), it is straightforward to show that the optimal contract that offers expected utility $\overline{V}$ is a fixed wage contract, with $V_n(\overline{V}) = \overline{V}$. Hence, given that the firm's gains from retaining against terminating

of $\beta = 1$ and that the firm cannot respond to the worker's outside offers.

the worker is strictly positive, we have $f^*(\overline{V}) = f^*(V_n(\overline{V})) = 0$. **Step 2.** Suppose $f^*(V_n(\underline{V})) > 0$. Then, the first order condition implies $V_n(\underline{V}) > \underline{V}$ (given that the marginal gains are strictly greater than the marginal losses at $V_n(\underline{V}) = \underline{V}$). Thus, at the beginning of a period before the labor market opens, all employed workers (who were retained and have survived from the prior period) would be promised an expected utility no less than $V_n(\underline{V})$. That is, all offers between $\underline{V}$ and $V_n(\underline{V})$ would be acceptable to all unemployed workers, but not any employed worker. But why would the firm offer a more expensive contract not expecting a higher probability of acceptance? This is a contradiction and so the desired result holds.

Proposition 6. *(i) Suppose $k = C_0 = 0$. Then all firms offer the same Walrasian wage $w = \theta$. (ii) Suppose either $k > 0$ or $C_0 > 0$. Then the highest wage offered in equilibrium, $c(\overline{V})$, is decreasing in k and C_0.*

One way to measure the size of the dispersion in the equilibrium wages offered is through the length of the interval $[0, c(\overline{V})]$, which includes all starting wages offered in the model's equilibrium, and remember 0 is the minimum equilibrium wage offered. Thus Proposition 6 says, on one metric, that, conditional on there be dispersion in the equilibrium wages offered, the dispersion is larger if the costs of job posting or termination are smaller.

Proposition 6 offers a testable prediction of the model, which is that labor market institutions and technologies that affect how costly job turnovers are for the firm have a impact on the distribution of the wages offered. Interestingly, however, the effect is not entirely monotonic. On the one hand, frictionless job turnover supports no pure wage dispersion. On the other hand, once job turnover is not entirely costless, then smaller costs of job termination or job posting always imply larger dispersion in the wages offered.

We will turn back to this important point later in the paper, as more analytical results and intuitions are developed in a different environment of the model.

3.3 Calibration: $\beta < 1$, Private Outside Offers

We calibrate the model to the U.S. data. For this, we differentiate the probability of receiving an offer for unemployed workers p_w^u from that for employed workers p_w^e. We also introduce an exogenous job separation probability $\lambda \in (0, 1)$. In additional, we assume that each unemployed worker receives an unemployment compensation of $b(> 0)$ units of consumption each period.[14] This compensation is financed by a payroll

[14] We take this as an approximation for the unemployment insurance that exists in practice. Observed unemployment insurance programs usually have more sophisticated benefits schemes (see for example Wang and Williamson (1996, 2002).

tax at a constant rate τ to balance the government's budget, period by period. These modifications make the model environment closer to the labor market the calibration is targeting, they however should not change the characterizations of the outcomes of the model.

The time period is set to be one month. The (monthly) interest rate is set to be 0.00417 to obtain an annual interest rate of 5%. The worker's discount factor is set to be $\beta = 1/(1 + 0.00417) = 0.9959$. We set $\delta = 0.0019$ so that the worker's expected (working) lifetime is 45 years.

The firm's period output is normalized to be $\theta = 1$. The worker's utility function is

$$u(c) = \frac{(1 + c)^{1-\eta}}{1 - \eta}, \ \forall c \geq 0,$$

where η is a positive constant. With this utility function, the relative risk aversion coefficient is $\eta c/(1+c)$, which is increasing in c.[15] Given that the worker's compensation is always less than the period output θ in equilibrium, $\eta c/(1 + c)$ is less than $\eta/2$. For the value of η then, we let $\eta = 4.5$ and $\eta = 6.5$ to obtain a lower and a higher value for the average coefficient of relative risk aversion, in two separate versions of the calibration.

We set $p^u_w = 0.43$ and $\lambda = 0.03$, following the estimates of Shimer (2007).[16] Given these, the equilibrium unemployment rate is calculated to be $u = 0.0711$ from the stationarity condition for u.[17]

It is straightforward to show that the calibration outcomes would be constant for all combinations of k and p_f with the same k/p_f ratio. This ratio measures, obviously, the expected costs of job posting each time a new vacancy is created. Given this, we will choose the ratio k/p_f, not the values k and p_f independently, as a free parameter in the calibration.[18]

Given the above, we choose the values for p^e_w, k/p_f, C_0 and b to target the following moments: (i) An aggregate replacement ratio of 41% from Shimer (2005). That is,

[15] We pick this IRRA utility function to bound the space of the worker's expected utility for the dynamic contract, in order to achieve better precisions in computation. This would not be necessary for the case of $\beta = 1$ (discussed in the next section), where it is possible to convert the states in expected utility to states in consumption, and that greatly reduces the size of the state space.

[16] In equilibrium, no vacant firms would post a contract which is not even acceptable to unemployed workers. Hence, p^u_w is equal to the U-E transition probability of 43%.

[17] Suppose the mortality probability is zero. Then the unemployment rate would be 6.5% as in Hornstein et al. (2011). This would not change our main results.

[18] In the literature, the probability of finding a worker (unemployed or employed) by vacant firms, p_f, is typically pinned down by the job opening rate, as in Wang and Yang (2015b). However, given that the number of jobs received by unemployed workers is $p^u_w u$ and the number of jobs received by employed workers is $p^e_w(1-u) \geq 0.022(1-u)$ (p^e_w must be greater than the E-E transition probability of 2.2%, as estimated in Nagypal (2008)), the job opening rate should be at least 5.5%, which is impossible to be consistent with the observed job opening rate of 3.4%, as estimated in Davis et al. (2006).

the unemployment benefit b is 41% of the average earned wage. (ii) An E-E transition probability of 2.2%, as estimated in Nagypal (2008). (iii) A mean-min ratio of 1.75, which is the median of the the estimates of Hornstein et al. (2007).[19] The values of the parameters chosen for the calibration are given in Table 1,[20] while the data and model measures of the targets are given in Table 2. Figures 1(a) and 1(c) display the density of the equilibrium distribution of wages offered,[21] for Calibrations 1 and 2, respectively. Figures 1(b) and 1(d) display the density for the equilibrium distribution of wages earned, again in the two versions of the calibration, respectively.

Table 1: Parameter values

	η	p_w^e	k/p_f	C_0	b	τ
Calibration 1	4.5	22%	0.222	0.16	0.3857	3.15%
Calibration 2	6.5	20%	0.222	0.47	0.3838	3.16%

Table 2: Calibration outcomes

Variable	Calibration 1	Calibration 2	Data	Source
U-E transition prob.	43%	43%	43%	Shimer (2007)
E-U transition prob.	3%	3%	3%	Shimer (2007)
E-E transition prob.	2.27%	1.54%	2.2%	Nagypal (2008)
Replacement ratio	41.13%	41.22%	41%	Shimer (2005)
Mean-min ratio	1.7444	1.7407	1.75	Hornstein et al. (2007)

Calibration 1 does a good job matching the targets. The calibrated model also generates an unimodal distribution for the starting wages offered. The distribution, however, is skewed to the right, not left. This is corrected in Calibration 2, where a larger value of η is chosen. With a larger η or smaller intertemporal substitutability of consumption for the worker, stronger motives for consumption smoothing force the optimal contract to move the worker's expected utility up at a slower rate, shifting the distribution of the employed workers' expected utilities to the left. This, in turn, induces vacant firms to offer lower starting wages, shifting the distribution of wages offered also to the left.

Calibration 2 falls short of matching the observed E-E (employment to employment) transition probability. This is natural, given there is no productivity heterogeneity in the model. In the data, a sizable part of job-to-job transitions occur when an employed worker gets a job offer from a more productive firm.

[19] In Hornstein et al. (2007), the estimated mean-min ratio is between 1.5 and 2.

[20] With $\eta = 4.5$, the worker who earns the median wage would have a coefficient of risk aversion of 2.18. The expected posting cost k/p_f, which is calibrated to be 0.222, is 0.213 in Shimer (2005).

[21] Note this is not the f^* in Proposition 5 which is for the expected utilities offered.

In both versions of the calibration, the model is able to replicate the observed mean-min ratio with reasonable E-E transition probabilities. This is in contrast with Hornstein et al. (2011), where in order to generate a mean-min ratio of 1.55, the E-E transition probability needs to be as high as 3.2% while the replacement ratio must be 0.

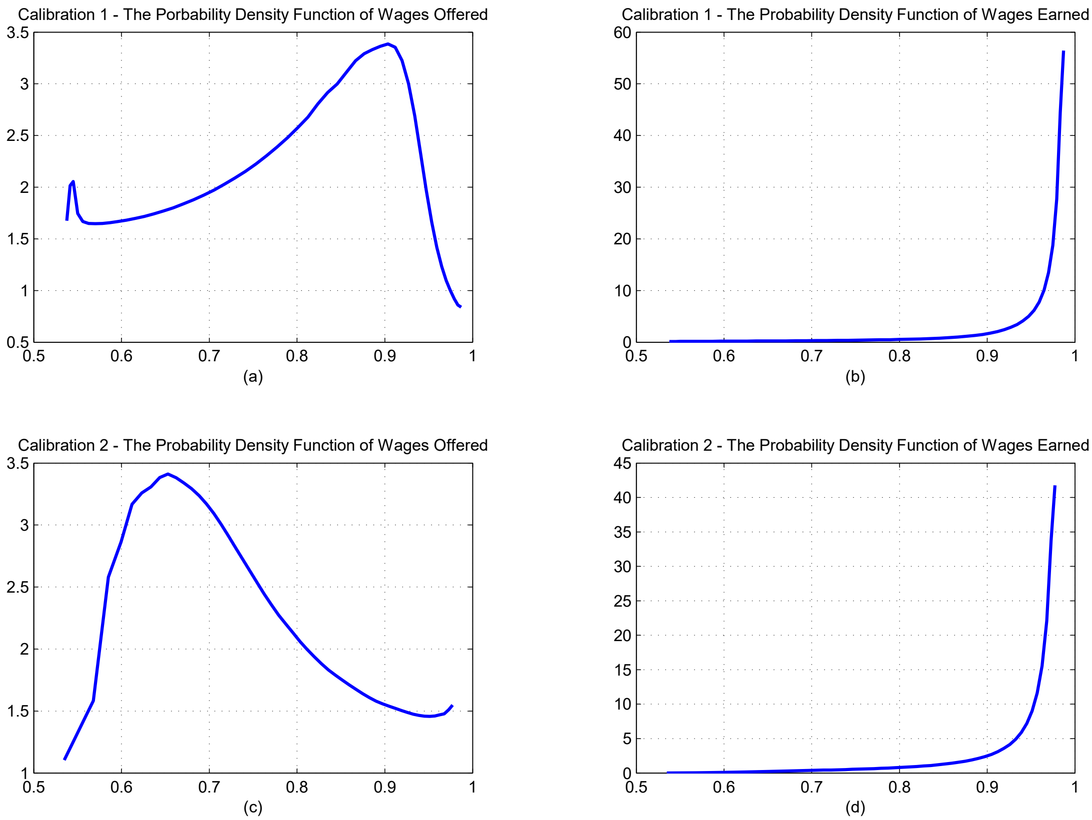

Figure 1: The equilibrium probability density functions of wages offered and earned.

The density of the distribution of wages offered resembles observed wage distributions in at least two ways. First, it has a long tail to the right. Second, it is unimodal, with a large peak in the middle of the support of the distribution.

The calibrated model also generates enough dispersion in the wages earned. Notice that the density of the equilibrium distribution of wages earned is consistent with optimal contracting, which prescribes rising expected utilities/wages for the employed worker. What is not desirable, obviously, is that the density is convex and moves sharply up for higher wages earned. There is, however, no extra force in the model to correct for this, by slowing down the speed of the rise of wage in tenure.

3.4 Summarizing Remarks

The analysis carried out so far indicates that a right amount of counteroffering is necessary for obtaining the right distributions of wages offered and earned. Unlimited counteroffering, which occurred with public outside offers, kills dispersion in the wages offered, while too little counteroffering, occurred in the case of private outside offers, gives a wrong distribution of the wages earned.

In search for the right amount of counteroffering, in the remainder of the paper we move on to study the model under the assumption that workers and firms do not discount future payoffs, by letting $\beta \to 1$. We show that with no discounting, the model has multiple equilibria where limited amounts of counteroffering are optimal, and by picking the right equilibrium with the right amount of counteroffering, the model, calibrated to the U.S. data, could then produce distributions of wages offered and earned that are consistent with the data. Moreover, assuming no discounting makes the comparison with Burdett and Mortensen (1998) and Burdett and Coles (2003) more direct, as none of them assume discounting. Assuming no discounting also allows for analytical characterizations of some of the key mechanisms for determining wage dispersion.

4 No Discounting ($\beta = 1$)

4.1 Public Outside Offers

Assuming publicly observed outside offers, we first characterize the optimal contract. We then characterize a stationary equilibrium of the model in a set of equations. Lastly, we study three examples of the model's equilibria.

4.1.1 Optimal Contracting

Proposition 7. *At $\beta = 1$ and with observable outside offers, it holds with the optimal contract that for all $V \in \Phi^*$ with $f^*(V) > 0$,*

(i) $V_r(\xi, V) = \max\{\xi, V\}$, for all $\xi \in \Phi^*$.

(ii) For all $\xi \in \Phi^*$, $I(\xi, V) = 1$ if $\xi < V$, and $I(\xi, V)$ is either 0 or 1 (equally optimal) if $\xi > V$.

(iii) $V_n(V) = V$ and $I_n(\xi, V) = 1$.

So in the limit as $\beta \to 1$, any time the worker's outside offer is below his current promised utility V (i.e., $\xi < V$), he is retained and moves into the next period with the same V. Suppose the outside offer exceeds his currently promised utility (i.e., $\xi > V$). Then the firm is indifferent between retaining the worker – by matching the outside offer – and terminating him.

The proof is constructed in three steps. **Step 1.** Remember for $\beta \in (0, 1)$, conditional on retention, the firm has no incentives to change the worker's expected utility unless he receives a superior outside offer; and this continues to hold in the limit as $\beta \to 1$. Specifically, if the worker's outside offer is strictly less than his current expected utility ($\xi < V$), then the self-enforcing constraint does not bind, perfect consumption/utility smoothing is achieved between the current and next periods.[22]

Step 2. We show it is optimal to retain the worker if the outside offer is below his current expected utility. Suppose $\xi < V$ and the firm wishes to retain the worker. Then, from Step 1, the continuation contract would offer the same expected utility V – which the firm promised the worker at the beginning of the period - to the worker from next period on. Note that $V \in \Phi^*$, for V is either the initial value offered, or that of a counteroffer – given Step 1. Suppose $\xi < V$ but the firm wishes to terminate the worker. Then upon termination it would return to the labor market to post a contract that offers expected utility say $V' \in \Phi^*$ for any new hire. Given V, $V' \in \Phi^*$ however, V and V' must give the firm the same value. Because of the costly process that follows each termination then, termination is inferior to retention.

Step 3. In the limit as $\beta \to 1$, if the worker receives a superior outside offer, then the firm should be indifferent between retaining – matching the outside offer – and terminating him. To see why, note the firm's value is $\beta\pi - (1-\beta)C_0$ if the worker is terminated and $U(\xi)$ if retained, ξ being the worker's outside offer and $\xi \geq V$, V in turn being his currently promised expected utility. Given $\beta = 1$, for all $\xi \in \Phi^*$ with $f^*(\xi) > 0$,

$$\beta\pi - (1-\beta)C_0 = \pi = \overline{U}(\xi) = \frac{-(1-\beta)k + p_f\gamma(\xi)U(\xi)}{1-(1-p_f\gamma(\xi))\beta} = U(\xi). \tag{29}$$

Obviously then, the indifference between retention and termination arises because, at $\beta = 1$, the costs that the firm incurs in terminating the current contract and starting the next is negligible (zero). This indifference, as we show later in the section, will result in multiple equilibria for the model, with degenerate and non-degenerate distributions of the wages/contracts offered.

Lastly, note also that for the worker, retained or terminated, his expected utility is ξ and so he is also indifferent between retention and termination.

4.1.2 The Limiting Stationary Environment

In the limiting stationary equilibrium, we define the firm's value function, as well as the stationarity conditions as follows.

The firm's value function For all $V \in \Phi^*$ with $f^*(V) > 0$, the expected value of a vacant firm that posts a contract (of the kind described above) offering expected

[22] On the contrary, if the worker's outside offers are private information, then the optimal contract is designed to address the trade-off between risk sharing and incentive provision (specifically, provide incentives for the worker to not only stay, but also report truthfully). Hence, even if the worker receives a worse outside offer such that the limited commitment constraint is not binding, the incentive compatibility constraint could be binding to dictate increasing the worker's wages automatically with tenure, as shown in Proposition 4.

utility V is

$$\pi = \overline{U}(V) = \frac{-(1-\beta)k + p_f\gamma(V)U(V)}{1-(1-p_f\gamma(V))\beta}. \tag{30}$$

This then implies that for any given $V \in \Phi^*$ with $f^*(V) > 0$, and all $\xi \in \Phi^*$ with $f^*(\xi) > 0$,

$$\begin{aligned} U(\xi) &= \frac{(1-\beta)k + [1-(1-p_f\gamma(\xi))\beta]\pi}{p_f\gamma(\xi)} \\ &= \frac{\beta(1-\beta)(\gamma(V)-\gamma(\xi))}{[1-(1-p_f\gamma(V))\beta]\gamma(\xi)}k + \frac{[1-(1-p_f\gamma(\xi))\beta]\gamma(V)}{[1-(1-p_f\gamma(V))\beta]\gamma(\xi)}U(V). \end{aligned} \tag{31}$$

Let $\underline{V} \equiv \inf\{V \in \Phi^* | \, f^*(V) > 0\}$ and $\overline{V} \equiv \sup\{V \in \Phi^* | \, f^*(V) > 0\}$. Then, for all $V \in \Phi^*$ with $f^*(V) > 0$, the expected value of a firm who employs a worker with a contract that promises expected utility V is

$$\begin{aligned} U(V) &= (1-\beta)(\theta - c(V)) + \beta\delta[\pi - (1-\beta)C_0] \\ &\quad + \beta(1-\delta)p_w \left\{ F^*(V)U(V) + \int_{\underline{V}}^{\overline{V}} I(\xi;V)U(\xi) + (1-I(\xi;V))[\beta\pi - (1-\beta)C_0]dF^*(\xi) \right\} \\ &\quad + \beta(1-\delta)(1-p_w)U(V), \end{aligned}$$

which, given (30) and (31), and by taking β to 1 and applying L'Hospital's rule, gives

$$\begin{aligned} &\left[(1-\delta) + \frac{\delta + (1-\delta)p_w(1-F^*(V))}{p_f\gamma(V)} - \frac{(1-\delta)p_w}{p_f}\int_{\underline{V}}^{\overline{V}} \frac{I(\xi;V)}{\gamma(\xi)}dF^*(\xi) \right] U(V) \\ = &\ (\theta - c(V)) - \left[\delta + (1-\delta)p_w \int_{\underline{V}}^{\overline{V}} 1 - I(\xi;V)dF^*(\xi) \right] C_0 \\ &- \frac{1}{p_f\gamma(V)} \left[\delta + (1-\delta)p_w \int_{\underline{V}}^{\overline{V}} 1 - I(\xi;V)\frac{\gamma(V)}{\gamma(\xi)}dF^*(\xi) \right] k, \end{aligned} \tag{32}$$

which gives the value of the firm for all $V \in \Phi^*$ at $\beta = 1$.

Stationarity conditions Two conditions must hold to make the equilibrium environment stationary. First, the measure of unemployed workers is time invariant:

$$1 - u = (1-\delta)[(1-u) + p_w u]. \tag{33}$$

Second, the distribution of employed workers is stationary: for all $V \in \Phi^*$,

$$(1-u)G(V) = (1-\delta)[1-p_w(1-F^*(V))](1-u)G(V) + (1-\delta)p_wF^*(V)u, \tag{34}$$

Moreover, under the above specified stationarity conditions, the probability with which a contract offering an expected utility $\xi \in \Phi^*$ is accepted by a randomly matched worker is

$$\gamma(\xi) = u + (1-u)\int_{\underline{V}}^{\xi}[1 - I(\xi; V)]dG(V). \tag{35}$$

4.1.3 Equilibrium Distributions

We now solve for the equilibrium distributions of contracts/wages offered that are consistent with the optimal contracts and the stationary values and distributions described above. To prepare for the analysis, note first that it is straightforward to show that $c(V)$ is strictly increasing in V. With this, we define

$$\underline{c} \equiv c(\underline{V}) \text{ and } \overline{c} \equiv c(\overline{V}) \tag{36}$$

to be, respectively, the lowest and highest wages offered. And, for all $c \in [\underline{c}, \overline{c}]$, let

$$\widetilde{\gamma}(c) \equiv \gamma(c^{-1}(c)),\ \widetilde{F}^*(c) \equiv F^*(c^{-1}(c)),\ \widetilde{G}(c) \equiv G(c^{-1}(c)), \tag{37}$$

$$\widetilde{I}(c'; c) \equiv I(c^{-1}(c'); c^{-1}(c)), \text{ with } c' \geq c. \tag{38}$$

With these, we can describe an equilibrium distribution of contracts offered in initial wages, instead of expected utilities, offered.

Because firms are free to enter and exit the market, in equilibrium it must hold that $\pi = 0$ which, given $\beta = 1$ and (30), implies that for all $V \in \Phi^*$ with $f^*(V) > 0$, $U(V) = 0$ which, given (32), holds if and only if for all $c \in [\underline{c}, \overline{c}]$ with $\widetilde{f}^*(c) > 0$,

$$\begin{aligned} \theta - c \;&=\; \left[\delta + (1-\delta)p_w \int_c^{\overline{c}} 1 - \widetilde{I}(c'; c)d\widetilde{F}^*(c')\right] C_0 \\ &\quad + \frac{1}{p_f \widetilde{\gamma}(c)}\left[\delta + (1-\delta)p_w \int_c^{\overline{c}} 1 - \widetilde{I}(c'; c)\frac{\widetilde{\gamma}(c)}{\widetilde{\gamma}(c')}d\widetilde{F}^*(c')\right] k, \end{aligned} \tag{39}$$

which requires that the firm breaks even on expected net profits – the value of outputs net of worker compensation and the costs related to worker termination and job posting is zero.

With the above, solving for a stationary equilibrium of the model is reduced to solving equations (1), (2), (33)-(35), (36)-(38) and (39) for p_w, p_f, u, $m, \widetilde{I}(\cdot,\cdot)$, $\widetilde{\gamma}(\cdot)$, $\widetilde{F}(\cdot)$ and $\widetilde{G}(\cdot)$.

From equation (39), if both C_0 and k are zero, then in equilibrium all firms offer the same Walrasian wage $c = \theta$. In other words, in order to have a non-degenerate distribution of wage offers, either the costs of terminating an existing job, C_0, or that of posting a new job, k, must be positive.

Why do the costs matter essentially for the wage dispersion? The answer is in equation (39), which formalizes the intuition of BM. Namely, it is the positive costs associated with job turnover that makes a higher wage offered worthwhile. A larger

c – a higher wage offered – lowers not only the firm's expected profits net of labor compensation, which is the left hand side of the equation, it also lowers the expected costs of job-turnover, which is on the right hand side of the same equation.[23]

In what follows, we look at three examples of the model's equilibria where the distribution of wages offered is analytically characterized. We start in Example 1 with the fixed wage contract studied in Burdett and Mortensen (1998). In Example 2, we let the firm to always retain the worker. In Example 3, we consider the equilibria where the contract is such that the worker is terminated if the outside offer is sufficiently high, and he is retained either the firm matches his outside offer, or he stays with the firm voluntarily with the current wage.[24]

Equilibrium 1: fixed wages (as in BM)

Suppose, as Proposition 7 prescribes to be optimal, the contract is such that the wage is fixed at a constant c and the worker is terminated whenever he receives a superior outside offer. That is, $\widetilde{I}(c';c) = 0$ for all $c' \geq c$. Note that this is the contract which was taken as given by Burdett and Mortensen (1998).

With c being the fixed wage, for all $c \in [\underline{c}, \overline{c}]$, (35) implies

$$\widetilde{\gamma}(c) = u + (1-u)\widetilde{G}(c) = \frac{\delta + (1-\delta)p_w}{\delta + (1-\delta)p_w(1-\widetilde{F}^*(c))}u, \tag{40}$$

where the second equality follows from (34). With this, (39) then implies that for all $c \in [\underline{c}, \overline{c}]$ with $\widetilde{f}^*(c) > 0$,

$$\theta - c = \left[\delta + (1-\delta)p_w\left(1 - \widetilde{F}^*(c)\right)\right]\left(C_0 + \frac{k}{p_f\widetilde{\gamma}(c)}\right). \tag{41}$$

As does equation (39), this only requires that the firm break even in the long-run; that is, what it earns from the employed worker is just enough to cover the costs of termination and job-posting.

The zero-value condition (41), by imposing a constraint on the relationship between each individual wage offer c and the distribution of the wages offered, $\widetilde{F}^*$, dictates a condition from which one could solve for $\widetilde{F}^*$. Specifically, given $\widetilde{I}(c';c) = 0$ for all $c' \geq c$, we have for all $c \in [\underline{c}, \overline{c}]$,

$$\delta + (1-\delta)p_w\left(1 - \widetilde{F}^*(c)\right) = \frac{-p_f\delta C_0 + \sqrt{(p_f\delta C_0)^2 + 4p_f\delta(\theta - c)k}}{2k},^{25}$$

[23] The literature has not discussed the role of the costs of termination and job posting in deriving a non-degenerate wage distribution. In BM and BC, the cost of termination is assumed to be null. BM does assume a non-negative k in their model, but that was not important for generating their wage distribution.

[24] Such a contract was shown, in Wang and Yang (2012), to be optimal in an environment that is partial but shares some key elements with the current model.

which, given (40) and $u[\delta + (1-\delta)p_w] = \delta$ by (33), in turn implies

$$\widetilde{F}^*(c) = 1 - \frac{1}{(1-\delta)p_w}\left\{\frac{p_f\delta}{2k}\left[-C_0 + \sqrt{{C_0}^2 + \frac{4k(\theta - c)}{p_f\delta}}\right] - \delta\right\}, \tag{42}$$

where $\underline{c} \in \mathbb{R}_+$, the lowest wage offered with $\widetilde{F}^*(\underline{c}) = 0$, and $\bar{c} \in \mathbb{R}_+$, the highest wage offered with $\widetilde{F}^*(\bar{c}) = 1$, are solved to be

$$\underline{c} = \theta - [\delta + (1-\delta)p_w]\left\{C_0 + \frac{[\delta + (1-\delta)p_w]k}{p_f\delta}\right\} \text{ and } \bar{c} = \theta - \delta\left(C_0 + \frac{k}{p_f}\right). \tag{43}$$

Furthermore, as in BM, in equilibrium the reservation wage for unemployed workers must be 0,[26] or $\underline{c} = 0$, which implies

$$\theta = [\delta + (1-\delta)p_w]\left\{C_0 + \frac{[\delta + (1-\delta)p_w]k}{p_f\delta}\right\}. \tag{44}$$

To summarize, the equilibrium values of p_w, p_f, u and m are given jointly by (1), (2), (33), and (44). And it is straightforward to show that if $M(1, v)$ is continuous in v with $\lim_{v\to 0} M(1, v)/v = 1$, an equilibrium exists if and only if $0 < \delta(C_0 + k) \leq \theta$. That is, if either of C_0 and k is positive and that their sum is not too large.[27]

Equation (43) gives the highest wage, $\bar{c}$, that allows an individual firm, who must take the equilibrium outcomes of the market as given, to break even on the offer he makes. At $\bar{c}$, the worker whom the firm employs would stay on the job until he dies - no other firm is making a higher offer to bid him away. So in order to break even at $\bar{c}$, what matters includes, obviously, only the variables on the right side of (43). In particular, $\bar{c}$ depends negatively on C_0 and k - all else equal a larger C_0 or k lowers directly the firm's profits and reduces its ability in offering a larger wage. Also, $\bar{c}$ depends positively on p_f - the probability with which a vacant firm is matched with a worker. Note that p_f is an endogenous variable whose value depends on C_0 and k. Last, a larger δ, which, all else equal, dictates larger job turnover, also lowers $\bar{c}$.

Suppose there is an increase in C_0 or k. How does that affect the equilibrium $\bar{c}$ and hence the support of the *equilibrium* wage offers - the interval $[0,\ \bar{c}]$ which, by one metric, measures the size of the dispersion of the wages offered? The answer is in equations (43) and (44). Let, for example, C_0 be larger. From (43), this lowers directly

[25] The case of $k = 0$ can be dealt with separately, or by letting $k \to 0$ in the above equation and in the following discussion, to yield the same outcomes.

[26] Suppose that workers would leave even for the same wage. Then, there is no mass point in the distribution of wages offered as shown in Burdett and Mortensen (1998), which, given that the reservation wage for unemployed workers is 0, implies $\underline{c} \geq 0$. Furthermore, if $\underline{c} > 0$, then a vacant firm could make a strictly greater profit by offering a wage strictly lower than $\underline{c}$ without sacrificing the probability of acceptance, but higher than the reservation wage of unemployed workers 0. Hence, we conclude $\underline{c} = 0$.

[27] Obviously, if $\theta < \delta(C_0 + k)$, then the firm would not be able to break even on the worker.

the level of $\bar{c}$. The larger C_0 also has an indirect - equilibrium - effect on $\bar{c}$. Specifically, given that p_w and p_f move in opposite directions, from (44) the equilibrium p_f must be higher,[28] and hence from (43) the equilibrium $\bar{c}$ must be higher. In other words, the direct negative effect of a larger C_0 on $\bar{c}$ is partly offset by the resulting positive equilibrium effect on $\bar{c}$ that follows.

But what gives the higher p_f that follows the increased C_0? The answer is that higher costs in worker turnover wash out firm profits, allowing fewer firms to survive in the new equilibrium that arises with the higher C_0 (or k).

Higher costs of job turnover alter not only the support but also the distribution of the firms and their starting wage offers over the support. This is shown in Figure 5, where the upper panel illustrates the two, and the only two, scenarios in which an increase in C_0 would affect the wage-offer density $\widetilde{f}^*$, and the lower panel depicts how an increase in k would affect that same object.[29]

What Figure 2 depicts can be explained as follows. As C_0 or k increases, the density function, now with a narrower support, is lifted up to let the same measure of firms be placed over a shorter interval in the starting wages they offer. In addition, two other effects take place simultaneously in determining the shape of the new equilibrium density $\widetilde{f}^*$:

Direct Effect: A larger C_0 or k, which increases the costs of termination and replacement, puts more pressure on firms that offer lower starting wages, giving them incentives to move to higher wage offers with lower worker turnover probabilities. This pushes individual firms to the *right* of the support of the distribution.

Equilibrium Effect: Similar to what was discussed for $\bar{c}$, a larger C_0 or k induces an equilibrium effect by way of increasing p_f - the probability with which a vacant job is filled. An increased p_f alleviates the effect that the (larger) costs C_0 and k impose on the firm, giving them incentives to move for lower starting wage offers. This pushes individual firms to the *left* of the support of the distribution.

In the end, of course, what matters is which of the above described effects is stronger. Obviously, from Figure 2, the direct effect is stronger in the case of an increasing k but a constant C_0, whereas the equilibrium effect is stronger where there is an increasing C_0 but a constant k.[30] Interestingly, in the case of $k = 0$, the two effects totally offset each other, leaving the density of the distribution in the starting wages offered constant in C_0.

[28] Suppose otherwise. That is, suppose p_f is lower. Then p_w would be higher. Then the equation (50) would be violated - its right hand side strictly greater than the left.

[29] A proof of why these are the relevant cases is given in Appendix H.

[30] To shed light on this, look back at equations (49)-(50) which determine the equilibrium $\bar{c}$. It can be shown that the direct effect dominates in the case of an increasing k and constant C_0, but there is not a definitive answer to which effect dominates in the case of a changing C_0 but a constant k.

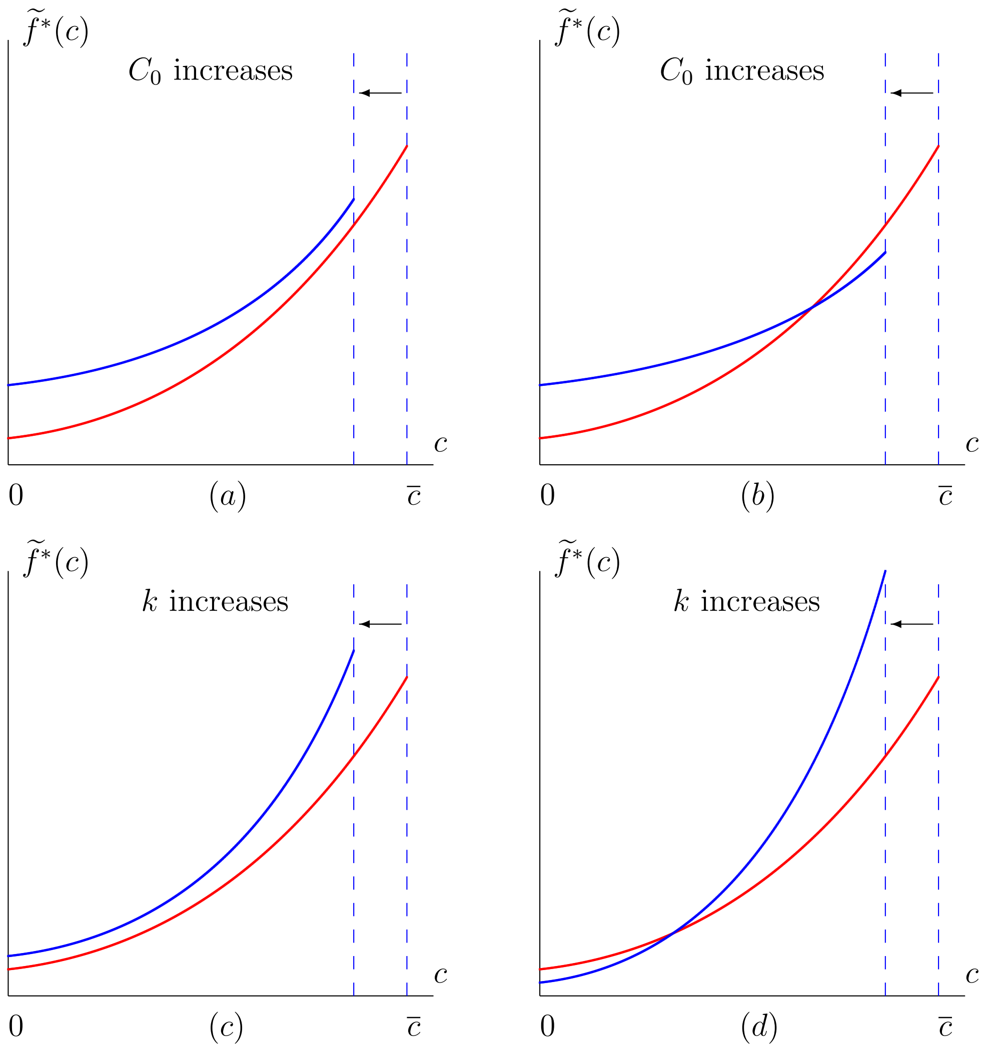

Figure 2: A larger C_0 or k alters the equilibrium distribution of the starting wages offered.

The following proposition summarizes some of the results just discussed.

Proposition 8. *Suppose* $0 < C_0 + k \leq \theta/\delta$. *(i) The density of the equilibrium wages-offered distribution is non-degenerate and convex whenever* $k > 0$. *(ii) The support of the equilibrium distribution of the wages offered, the interval* $[0, \bar{c}]$*, shrinks (i.e.,* $\bar{c}$ *decreases) as* C_0 *or* k *increases. (iii) Suppose* $k = 0$ *(or* $k \to 0$*). Then the equilibrium distribution of the wages offered is non-degenerate and uniform, with*

$$\widetilde{F}^*(c) = \frac{c}{\theta - \delta C_0}, \ \forall c \in [0, \theta - \delta C_0]. \tag{45}$$

To close this part of the discussion, we mark two points from the analysis. First, the distribution of the wages offered is degenerate if and only if the costs of worker turnover are zero (i.e., $C_0 = k = 0$). Second, once this condition is violated, larger costs of worker turnover always imply a smaller dispersion in the wages offered in the model's equilibrium. (Observe the non-monotonicity in the relationship between dispersion and the turnover costs!)

The first point says that the costs of worker turnover are essential for the existence of the pure dispersion we are looking for. Moreover, the stringent condition on C_0 and k (they must both be zero) then implies that a non-degenerate wage dispersion might be a more realistic description of what one should expect from real world labor markets, where some costs and frictions inevitably exist.

The second point offers a testable prediction on wage dispersion: there should be less dispersion in the wages offered in markets where the environment - regulations and technologies in particular - imposes larger costs of hiring and termination on firms. For example, firing taxes could reduce wage dispersion. For another example, the internet, which supports less expensive job advertising, could have increased, rather than decreased, wage dispersion.

In this equilibrium, it also holds, as is proved in Appendix J, that the mean of the wages offered is decreasing in C_0 and k. In addition, numerical examples we computed suggest that the variance of the wages offered is also decreasing in C_0 and k. With these, the model indicates that larger costs of worker turnover, resulting from a firing tax for example, while reducing wage inequality, may depress rather than improve worker welfare, whereas policies and technologies that reduce these costs may make workers better off.[31]

Equilibrium 2: matching all superior outside offers

Instead of offering a constant wage and never responding to any of the worker's outside offers, the firm in this equilibrium counters any outside offer of the worker that is superior to his ongoing value. That is, $\widetilde{I}(c'; c) = 1$ for all $c' \geq c$.

Because no offer would be accepted by an employed worker, (35) implies

$$\widetilde{\gamma}(c) = u, \forall c \in [\underline{c}, \overline{c}],$$

with which the zero-profit condition (39) writes

$$\theta - c = \delta \left(C_0 + \frac{k}{p_f u} \right).$$

So the equilibrium distribution is degenerate; and, following the same argument used in the proof of Proposition 1, the equilibrium must be such that all workers are paid the same monopsony wage $c = 0$.

[31] In this equilibrium of the model, a larger C_0 or k not only reduces average starting wages paid, with fewer firms that could survive the larger turnover costs, it also increases the economy's unemployment rate.

Equilibrium 3: limited-counteroffers

In the two equilibria discussed above, the firm either responds to no outside offers, or matches all of the worker's superior outside offers. We now consider cases in between, letting the firm match the worker's outside offer up to a pre-specified upper limit which depends positively on his current wage. For tractability, assume $k = 0$. Later in the quantitative analysis of the model, there will be positive C_0 and k.

Consider an equilibrium of the model where for all c, $c' \in [\underline{c}, \overline{c}]$ with $\widetilde{f}^*(c) > 0$,

$$\widetilde{I}(c'; c) = \begin{cases} 1, & \text{if } c' \leq m(c) \\ 0, & \text{if } c' > m(c) \end{cases}, \tag{46}$$

where $m(c)$ is taken to be given, continuous and strictly increasing in c, and with $c \leq m(c) \leq \overline{c}$. That is, at each current wage c the firm is willing to counter outside offers up to and only up to $m(c)$. Note that, to the firm, given its indifference between retention and termination at any c' with $c' \geq c$, any choice of the function $m(\cdot)$ that meets the above specified condition is optimal.

Obviously, letting $m(c) = c$ for all c gives Equilibrium 1, and letting $m(c) = \overline{c}$ for all c gives Equilibrium 2. To think about the cases in between, we let $m(\cdot)$ be such that $m(c) < \overline{c}$ at least for some c (or because $m(\cdot)$ is monotonic, $m(c)$ is strictly less than $\overline{c}$ if c is below a cutoff). Note that such a policy towards outside offers leaves at least some (larger) outside offers not matched.

Given $k = 0$ and (46), for all $c \in [\underline{c}, \overline{c}]$ with $\widetilde{f}^*(c) > 0$, the zero-profit condition $\pi = 0$, or (39), is written as

$$\theta - c = \left[\delta + (1-\delta)p_w\left(1 - \widetilde{F}^*(m(c))\right)\right] C_0, \tag{47}$$

where $p_w\left(1 - \widetilde{F}^*(m(c))\right)$ measures the employed worker's probability of job turnover. Assuming $C_0 > 0$, the above equation gives

$$\widetilde{F}^*(m(c)) = 1 - \frac{1}{(1-\delta)p_w}\left(\frac{\theta - c}{C_0} - \delta\right),$$

which, given that $m(\cdot)$ is strictly increasing, in turn implies that for all $c \in [m(\underline{c}), \overline{c}]$ with $\widetilde{f}^*(c) > 0$,

$$\widetilde{F}^*(c) = 1 - \frac{1}{(1-\delta)p_w}\left(\frac{\theta - m^{-1}(c)}{C_0} - \delta\right). \tag{48}$$

We now solve for $\underline{c}$, $\overline{c}$ and p_w respectively. We first show $\underline{c} = 0$, in 3 steps. **Step 1:** $V_0 \leq \underline{V}$. This holds because no worker, employed or unemployed, would accept a job that offers an expected utility below V_0, which in turn should not be offered in the first place. **Step 2:** $V_0 = \underline{V}$. As in Burdett and Mortensen (1998), a vacant firm offering the minimum expected utility $\underline{V}$ in Φ^* (or the minimum wage $\underline{c}$) would not have its offer accepted by any employed worker.[32] That is, any offer of $\underline{V}$ could be accepted only by an unemployed worker. Now suppose $V_0 < \underline{V}$. Then the firm who offers $\underline{V}$ is better off offering V_0 instead, for both offers would be accepted with the same probability (by an unemployed worker) but V_0 gives the firm a strictly higher value.[33] **Step 3:** $\underline{c} = 0$. Since the probability of receiving an offer is the same for both unemployed and employed workers, unemployed workers would accept any job paying no less than what is earned while staying unemployed, which is zero. That is,

$$V_0 = \underline{V} = u(\underline{c}) + (1-\delta)\left[p_w \int_{\Phi^*} \max\{\xi, \underline{V}\} dF^*(\xi) + (1-p_w)\underline{V}\right],$$

which, given (3) and $\beta = 1$, in turn implies $\underline{c} = 0$.

We now solve for $\bar{c}$. Given $\widetilde{F}^*(\bar{c}) = 1$ and $m(\bar{c}) = \bar{c}$,[34] (48) implies

$$1 = \widetilde{F}^*(\bar{c}) = \widetilde{F}^*(m(\bar{c})) = 1 - \frac{1}{(1-\delta)p_w}\left(\frac{\theta - \bar{c}}{C_0} - \delta\right), \tag{49}$$

which then gives $\bar{c} = \theta - \delta C_0$.

To solve for p_w, notice that given $\underline{c} = 0$, (48) implies

$$\widetilde{F}^*(m(0)) = 1 - \frac{1}{(1-\delta)p_w}\left(\frac{\theta}{C_0} - \delta\right),$$

which in turn gives

$$p_w = \frac{1}{1 - \widetilde{F}^*(m(0))} \frac{\theta - \delta C_0}{(1-\delta)C_0}. \tag{50}$$

With the above, (48) now gives

$$\widetilde{F}^*(c) = \widetilde{F}^*(m(0)) + \left(1 - \widetilde{F}^*(m(0))\right) \frac{m^{-1}(c)}{\theta - \delta C_0}, \; \forall c \in [m(0), \theta - \delta C_0]. \tag{51}$$

Note also that given that $m(c)$ is assumed to be continuous and strictly increasing, the distribution function $\widetilde{F}^*$ is also continuous and strictly increasing on $[m(0), \theta - \delta C_0]$, having no mass point on $(m(0), \theta - \delta C_0]$.

The counteroffer policy $m(\cdot)$ (more specifically $m(0)$) divides the labor market into two segments, the first of which including firms offering a wage $c \in [0, m(0)]$ to target unemployed workers – these offers would be countered if they are received by an employed worker. These firms, following the logic of Diamond (1971), will offer uniformly the monopsony wage $c = 0$. The second segment of the market then includes firms offering a non-degenerate distribution of wages $c \in (m(0), \theta - \delta C_0]$ to target both employed and unemployed workers, as in Burdett and Mortensen (1998).

[32] Remember we assume that any employed worker would not take a new job that offers a value equal to his current job.

[33] Specifically, the firm could increase its profit by decreasing the starting wage, while still remaining acceptable to unemployed workers.

[34] Note that $c \le m(c) \le \bar{c}$ for all c implies $m(\bar{c}) = \bar{c}$.

Given $m(\cdot)$ and given $\widetilde{F}^*(m(0))$, we have solved, in equation (51), for the non-degenerate part of the distribution of wage offers. What is not yet determined is $\widetilde{F}^*(m(0))$, the relative size of the market segment where the monopsony wage is offered. The value of $\widetilde{F}^*(m(0))$, however, is not uniquely determined. In fact, the model has an equilibrium for each combination of p_w and $\widetilde{F}^*(m(0))$ that satisfy (50). In other words, the model has a continuum of equilibria that differ in the size and composition of the market. To see this, substitute $p_w = M(1, v)$ and (56) into (47) to rewrite the zero profit condition for a firm offering starting wage c as

$$\theta - c = \left[\delta + (1-\delta)M(1,v)(1-\widetilde{F}^*(m(0)))\left(1 - \frac{c}{\theta - \delta C_0}\right)\right] C_0, \tag{52}$$

From the above, the indeterminacy of the model's equilibrium arises because for each $\widetilde{F}^*(m(0))$, free entry and exit of firms would ensure a "right" size of the market m and a "right" measure of vacant firms v in the market to make the product of $M(1, v)$ and $1-\widetilde{F}^*(m(0))$ such that the zero profit condition is met for each starting wage c offered with a positive probability. Note that there is a one-to-one relationship between the equilibrium size of the market, m, and the equilibrium measure of vacant firms in the market, v. Specifically, in equilibrium

$$m = v + (1-u) = v + \frac{(1-\delta)M(1,v)}{\delta + (1-\delta)M(1,v)}.$$

Hence, given that $M(1, v)$ is strictly increasing in v, m is also strictly increasing in v.

Equation (52) also indicates that for an equilibrium with a larger labor market (larger equilibrium m or v), a larger fraction of the vacant firms would be offering a monopsony rather than a non-monopsony wage. This intuition goes as follows. (a) A larger fraction of vacant firms offering the monopsony wage (i.e., a larger $\widetilde{F}^*(m(0))$) makes it less likely for an incumbent firm to lose a worker to an outside offer. This reduces worker turnover and increases the firm's expected profits, inducing more firms to enter the market and expanding the size of the market. (b) A larger v (or m) implies both a larger p_w and a smaller p_f. That is, a larger market makes it harder for all vacant firms to find a worker but easier for the incumbent firms to lose an existing worker (holding the distribution of wages offered constant), reducing the expected profits of firms. On the one hand, this deters firms from entering the market. On the other hand, the costs from the just mentioned faster worker turnover fall more heavily on the firms who offer the non-monopsony wage. These firms, by targeting both employed and unemployed workers, get their vacancy filled more quickly and are thus subject to more frequent worker turnover. This reduces the value of offering a non-monopsony wage, inducing a larger fraction of vacant firms to offer the monopsony wage.

Obviously, for the above described equilibrium to exist, it must hold that

$$0 \leq \widetilde{F}^*(m(0)) \leq \frac{C_0 - \theta}{(1-\delta)C_0}, \tag{53}$$

so that p_w, given in (50), is between 0 and 1. Hence, it must hold that $\theta \leq C_0 \leq \theta/\delta$.[35]

Observe, importantly, that equation (51) shows that as long as $m(0) < \bar{c}$, then the equilibrium distribution of wage offers is not degenerate. Moreover, the higher is $m(0)$, the smaller is the interval $[m(0), \bar{c}]$, and thus the smaller is the dispersion of the wages offered. Put differently, if the firm responds more aggressively to its worker's outside offers, then there is smaller dispersion in the wages offered.

Observe, lastly, that the above conclusion is consistent with the results obtained under $\beta < 1$, where the case of publicly observed outside offers corresponds to the case of $\beta = 1$ with $m(0) = \bar{c}$ so the firm is free to counter any outside offer; and the case of private outside offers corresponds to the case here with $m(c) = c$, for all c, so the firm is not able to counter any outside offer.

What kinds of wage offer distributions would the above constructed equilibria produce? Obviously, the model admits many possibilities. As an example, suppose the function m is convex. That is, as wage goes up, firms become increasingly more aggressive in matching the worker's outside offers. Then, as is straightforward to show, the resulting distribution of wage offers $\widetilde{F}^*$ is concave by (51), which implies that the density for the starting wages offered, $\widetilde{f}^*$, is a decreasing function. That is, higher wages are offered by fewer firms.

4.2 Private Outside Offers

Although the non-degenerate distribution of the fixed wage contracts offered is derived under the assumption of publicly observed outside offers, the fixed wage contracts remain incentive-compatible, and hence optimal, if the outside offers are private to the worker and not observed by the firm. Given this, what was obtained in the above section, where wages are fixed in time, as in BM, continues to be an equilibrium of the model if outside offers are private to the worker.

In fact, this is the unique equilibrium in the case of private outside offers. To see this, notice first that among all the optimal contracts in the case of public outside offers (where a truth-telling constraint need not be imposed in obtaining optimality), the fixed wage contract is the only contract that would induce truth-telling if outside offers were private. This then implies that the fixed wage contract must also be the only optimal contract in the case of private outside offers (where a truth-telling constraint

[35] Suppose $\theta > C_0$. Then, given $k = 0$, a vacant firm that offers a fixed wage of zero, which would be accepted by any unemployed worker matched, can earn a strictly positive profit, even if it keeps the worker for only one period. This could not be an equilibrium. Suppose $\theta < \delta C_0$. Then no firms can earn a non-negative profit, for the output is not enough to cover the expected cost of termination in each period. Note that the probability for an incumbent firm to lose the worker is at least δ.

must be added in deriving the optimum). Thus given any equilibrium that may exist in the case of private outside offers, the fixed wage contract must be the optimal contract. And last, the fixed wage contract does support a stationary equilibrium which, given the above, must then be the unique equilibrium of the model, for the case of private outside offers.

4.3 BM and BC Again

With the same assumption of $\beta = 1$ and the same model, the analysis presented above differ substantially from that in Burdett and Mortensen (1998) and Burdett and Coles (2003). The difference starts with how the firm's optimization is formulated. In Burdett and Mortensen (1998) and Burdett and Coles (2003), firms are assumed to choose a V to maximize their *steady state profit*, defined as the product of the probability of offer acceptance and the expected profit per new hire. In our discrete time setup, this would be equivalent to maximizing

$$p_f[u + (1-u)G(V)]\frac{\theta - c(V)}{\delta + (1-\delta)p_w(1-F(V))}, \tag{54}$$

where $p_f[u + (1-u)G(V)]$ is the probability with which a job that offers expected utility V is accepted, and $1/[\delta + (1-\delta)p_w(1-F(V))]$ measures how long the job, once accepted, would last, and $\theta - c(V)$ is the net profit that the job generates while it lasts.[36]

Obviously, this differs from the value $U(V)$ in our analysis that the firm seeks to maximize, which, given in (32), is derived directly by taking the discount factor β to 1. In particular, notice that the cost parameters, C_0 and k, which appear in (32) but not (54), affect the equilibrium outcomes in our analysis, but not theirs.

The objective function (54) would not have been correctly formulated. The objective should be the mean of the flow profit that the firm achieves over all the states it would possibly visit in its infinite life, including the states in which it is vacant, and the states in which a worker is currently employed. Unlike (54), such a formulation would account for the costs that the firm incurs in terminating an existing job (i.e., C_0) and in posting a new job (i.e., k), as in done in our analysis.

4.4 Calibration: $\beta = 1$, Public Outside Offers

Remember with publicly observed outside offers, the model has multiple equilibria. To calibrate the model, we fix the equilibrium at the one described in Example 3, where

[36] Note that (54) corresponds exactly to the objective in Burdett and Mortensen (1998) but not Burdett and Coles (2003). The discrete time formulation of the firm's objective in Burdett and Coles (2003), although in the same spirit, differs from (54) because of the dynamics in the wage-tenure contract.

the optimal contract pays a fixed wage to the worker but stands ready to counter any outside offer up to a threshold in expected utility. Differing from in the case of $\beta < 1$, here we use the more standard CRRA utility function:

$$u(c) = \frac{c^{1-\eta}}{1-\eta},\ \forall c \geq 0,$$

where $\eta(> 0)$ being the coefficient of constant relative risk aversion.

As for the case of $\beta < 1$, a period is one month, the interest rate is 0.00417, and the worker's mortality rate is $\delta = 0.0019$. Period output is again normalized at $\theta = 1$. We set $p_w^u = 0.43$ and $\lambda = 0.03$ to follow Shimer (2007). We set the equilibrium unemployment rate at $u = 0.0711$ for meeting the stationarity condition. We then choose the probability of receiving an offer for employed workers p_w^e, the expected posting cost k/p_f, the termination cost C_0, as well as the the firm's counteroffer policy $m : [\underline{c}, \overline{c}] \to [\underline{c}, \overline{c}]$[37] to target the following: (i) an E-E transition probability of 2.2%; (ii) a mean-min ratio of 1.75; and (iii) a truncated log-normal distribution for starting wages.[38] More specifically, to generate a truncated log-normal distribution of the starting wages offered, we set the scale parameter at $\sigma = 0.25$ and choose the location parameter μ to be such that the mode of the distribution is at $0.4\underline{c} + 0.6\overline{c}$.[39] We then choose the unemployment benefit b to target an equilibrium average replacement ratio of 41%, and the payroll tax rate τ to balance the government's budget period by period. Last, we choose the value of η to be such that the unemployed worker is indifferent between accepting a job that offers the lowest starting wage $\underline{c}$ and staying unemployed.[40]

Table 4: Parameter values

η	p_w^e	k/p_f	C_0	b	τ
2.61	19%	0.1333	0.4	0.3639	3.14%

Table 5: Calibration outcomes

[37]That is, given the worker's current wage c, the firm would match an better outside offer up to $m(c) \in [c, \overline{c}]$.

[38]Mortensen (2003) shows that the log-normal distribution approximates the observed distribution of starting wages quite well. Given that the distribution in our model has a bounded domain of $[\underline{c}, \overline{c}]$, our model is not good for generating a log-normal distribution which has an unbounded interval $[0, \infty)$.

[39]The calibration outcomes are quite robust in the scale parameter σ. If we pick a larger σ, however, the variance of starting wages offered (accordingly the variance of wages earned) would indeed be larger.

[40]Given $p_w^e = 0.19 < 0.43 = p_w^u$, if workers are not sufficiently risk averse (i.e., if η is too small), then they might want to reject a job that offers the lowest starting wage. In other words, η must be sufficiently large in order to make the unemployed worker willing accept a job that offers the lowest starting wage.

Variable	Model	Data	Source
U-E transition prob.	43%	43%	Shimer (2007)
E-U transition prob.	3%	3%	Shimer (2007)
E-E transition prob.	2.31%	2.2%	Nagypal (2008)
The replacement ratio	41%	41%	Shimer (2005)
The mean-min ratio	1.7439	1.75	Hornstein et al. (2007)

Tables 4 gives the values of the parameters chosen in the calibration. Table 5 gives the outcomes that the calibrated model generates, in comparison with the targets. Figures 3(a) and 3(b) depict the equilibrium distribution of wages offered, $\widetilde{f}^*(\cdot)$, and that of wages earned, $g^*(\cdot)$ respectively. Figure 3(c) depicts the firm's equilibrium counteroffer policy.

The calibrated model does well in matching the targets, including the observed mean-min ratio. More importantly, the calibrated model, relative to that with $\beta < 1$, does much better in generating a density, for both the distribution of wages offered and the distribution of wages earned, that looks similar to the data. What the model falls short of generating is a sufficiently long tail in both the distributions of wages offered and earned to better resemble the observed log-normal distributions in the data. But this may just be what the model, with identical workers and homogeneous firms, could be expected to achieve.[41]

From Figure 3(c), in the calibrated model the firm makes more aggressive counteroffers for workers whose expected utility is in the middle of the contract's state space. Observe also that in the equilibrium of the calibrated model, the minimum wage offered, $\underline{c} = 0.5090$, is larger than the unemployment benefit $b = 0.3639$. This is intuitive, for the job that offers a low starting wage also offers a monotonically increasing wage profile.

5 Wage Dispersion: Europe vs. U.S.

One of the most important labor-market developments over the recent decades – since the 1980s – is the increase in wage differentials in the U.S. relative to the much smaller and decreasing wage dispersion in continental Europe.[42] For the U.S., the increase is the case for both between-group and within-group wage differentials with respect to education and experience.

There is a large literature that explains the increasing between-group wage disper-

[41]To improve on this dimension of the calibration, one might suggest to pick a smaller value for the location parameter μ to push the mode of the distribution to the left side of the domain. We failed, however, to find such a μ and a counteroffer policy that are consistent with the outcomes desired.

[42]See Bertola and Ichino (1995), Gottschalk and Smeeding (1997) and Acemoglu and Autor (2011).

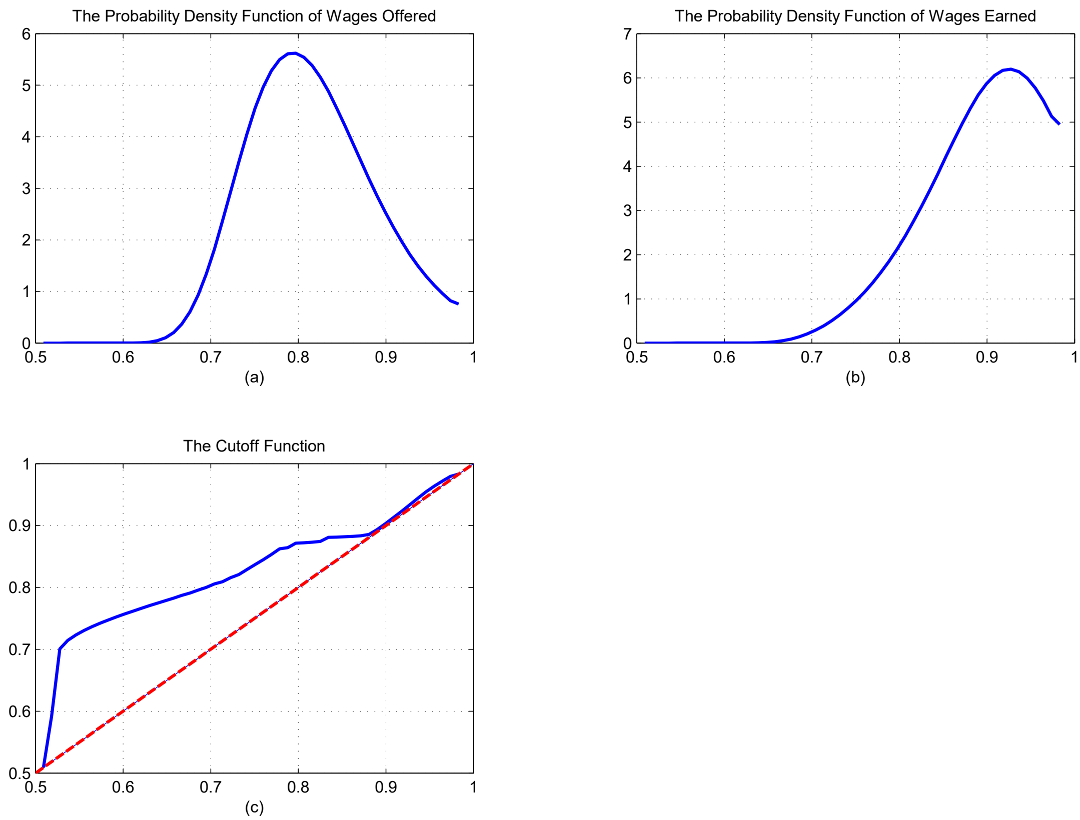

Figure 3: The equilibrium probability density functions of wages offered and earned, and the cutoff function.

sion,[43] but little has been said about why the U.S. also has a larger and increasing within-group wage dispersion. There is another literature that seeks to explain what happened in Europe, much of which views the sharp comparison with the U.S. as a result of institutional differences. As is well known, the labor markets in continental Europe (Germany and France in particular) are characterized by many institutional rigidities, with heavier costs associated with employment terminations, in the form of firing taxes, or other types of costs that employment protection would impose on the firm, whereas the U.S. is the standard example of flexibility and freedom of the market. Yet, much of this literature ultimately resorts to human capital, productivity, and labor unions for closing the story.

The pure theory offers a novel and potentially powerful perspective on the comparison. Could the rising inequality in the U.S. be resulting from an efficiency improvement in the labor market technologies (i.e., a reduced k in the model), as more vacancies

[43]The literature focuses especially on evaluating the skill/college premium. For example, skill-biased technological changes and international trade with less developed countries where unskilled labors abound, increase the relative demand for skilled workers in the U.S., which in turn would increase the skill premium (see Richardson, 1995; and Acemoglu, 2002).

had been posted online instead of on the newspaper, and more informational exchanges between firms and job searchers had been channeled through wireless or internet platforms at greatly reduced human and financial costs. Could the Europe - U.S. gap in wage inequality be due to the large policy induced termination costs in Europe (i.e., a larger C_0), which reduced the pure wage dispersion in Europe? Answering these questions would amount to testing empirically two specific predictions of our model which, being beyond the scope of our current study, could be hoped to offer new insights on the much discussed data.

6 Concluding Remarks

In this paper, we have looked for a pure theory of wage dispersion using the model of Burdett and Mortensen (1998), with identical workers and homogeneous firms. Differing from existing studies that put restrictions on how firms could react to their worker's outside offers, we let job offers be dynamic contracts with which firms respond optimally to the worker's outside offers. We show, analytically and quantitatively, that this is important for obtaining a pure theory of wage dispersion that makes sense with labor market observations.

Macroeconomists have pursued the idea of a pure theory of wage dispersion for long time. This is the first time such a theory is calibrated somewhat successfully to the data. The message which we think this analysis delivers is that the pure theory for wage dispersion should and can be taken more seriously, both theoretically and quantitatively, for interpreting labor market data and evaluating labor market policies.

To close the discussion, we note that, although the analysis has been carried out in a setting that is kept as close to Burdett and Mortensen (1998) as possible, requiring the discount factor be one is not the only way to achieve the right amount of counteroffering and hence the right wage distributions in the model's equilibrium. Imagine, for example, that in the case of $\beta < 1$ the firm could verify the worker's outside offer at a cost. The magnitude of this cost could then be chosen as an extra free parameter for producing the right wages offered and earned distributions. Even in the case of $\beta = 1$, an independent element could easily be incorporated into the model to determine the threshold for termination in the calibration. We leave these, and many other possibilities of extending this current work, for future research to explore.

Appendix

A Proof of Proposition 1

Lemma 1. $\Phi = [V_0, V_{\max})$.

Proof. The proof takes two steps.

Step 1 We show that if $V \in \Phi$, then $V \geq V_0$. Suppose $V \in \Phi$. Then there exists a feasible contract $\{c(V'), I(\xi; V'), V_r(\xi; V'), I(V'), V_n(V') : \xi \in \Phi^* \text{ and } V' \in \Phi\}$ satisfying (7)-(14) such that

$$\begin{aligned} V &= u(c(V)) + \beta(1-\delta)p_w \int_{\Phi^*} I(\xi; V)V_r(\xi; V) + (1 - I(\xi; V)) \max\{\xi, V_0\} dF^*(\xi) \\ &\quad + \beta(1-\delta)(1-p_w)[I(V)V_n(V) + (1 - I(V))V_0] \\ &\geq u(0) + \beta(1-\delta)\left[p_w \int_{\Phi^*} \max\{\xi, V_0\} dF^*(\xi) + (1-p_w)V_0\right] \\ &= V_0 \end{aligned}$$

where the first equality follows from (7), the inequality follows from $c(V) \geq 0$ by (8), $V_r(\xi; V) \geq \max\{\xi, V_0\}$ for all $\xi \in \Phi^*$ with $I(\xi; V) = 1$ by (11), and $V_n(V) \geq V_0$ by (14), and the last equality follows from (3).

Step 2 We show that there exists a feasible contract satisfying (7)-(14) that attains all $V \in [V_0, V_{\max})$. This contract is constructed as follows: For all $\xi \in \Phi^*$ and all $V \in [V_0, V_{\max})$,

$$I(\xi; V) = 1,\ V_r(\xi; V) = \max\{\xi, V\},\ I(V) = 1, \text{ and } V_n(V) = V$$

where $c(V) \geq 0$ is set to be such that (7) holds. It is straightforward to show that such $c(V)$ exists and the lemma is then proven. ■

By Lemma 1, constraints (10) and (13) can be simply ignored. Furthermore, constraints (9) and (12) are replaced by

$$I(\xi)(1 - I(\xi)) \geq 0,\ \forall \xi, \tag{55}$$

$$I(1 - I) \geq 0, \tag{56}$$

respectively. That is, we allow for stochastic termination. This is for technical convenience since the optimal termination is later shown to be deterministic, instead of stochastic.

Let α, μ, $\beta(1-\delta)p_w\lambda(\xi)$, $\beta(1-\delta)p_w\eta(\xi)$, $\beta(1-\delta)(1-p_w)\lambda_n$ and $\beta(1-\delta)(1-p_w)\eta_n$ be the Lagrangian multipliers for (7)-(9), (11)-(12), and (14) respectively. Then, the

Kuhn-Tucker conditions are

$$-(1-\beta)+\alpha u'(c)+\mu=0, \tag{57}$$

$$f^*(\xi)L(\xi)+\lambda(\xi)(1-2I(\xi))=0,\ \forall\xi, \tag{58}$$

$$f^*(\xi)I(\xi)(U'(V_r(\xi))+\alpha)+\eta(\xi)=0,\ \forall\xi, \tag{59}$$

$$L_n+\lambda_n(1-2I)=0, \tag{60}$$

$$I(U'(V_n)+\alpha)+\eta_n=0, \tag{61}$$

$$\mu c=0, \tag{62}$$

$$\lambda(\xi)I(\xi)(1-I(\xi))=0,\ \forall\xi, \tag{63}$$

$$\eta(\xi)(V_r(\xi)-\max\{\xi,V_0\})=0,\ \forall\xi. \tag{64}$$

$$\lambda_n I(1-I)=0, \tag{65}$$

$$\eta_n(V_n-V_0)=0, \tag{66}$$

$$\mu,\lambda(\xi),\eta(\xi),\lambda_n,\eta_n\geq 0,\ \forall\xi. \tag{67}$$

where

$$L(\xi)\equiv U(V_r(\xi))-[\beta\pi-(1-\beta)C_0]+\alpha(V_r(\xi)-\max\{\xi,V_0\}), \tag{68}$$

$$L_n\equiv U(V_n)-[\beta\pi-(1-\beta)C_0]+\alpha(V_n-V_0). \tag{69}$$

Furthermore, the Envelope Theorem gives

$$(\Gamma U)'(V)=-\alpha. \tag{70}$$

Lemma 2. *Suppose that U is decreasing and concave with $U(V_0)>\beta\pi-(1-\beta)C_0$ and $U'(V_0)=0$. Then,*

(i) There exists $\overline{\xi}\in[V_0,V_{\max}]$ such that for all $\xi\in\Phi^*$,

$$I(\xi)=\begin{cases}1 & \text{, if } \xi<\overline{\xi}\\ 0 & \text{, if } \xi>\overline{\xi}\end{cases};$$

(ii) $I=1$;

(iii) For all $\xi\in\Phi^*$, $V_r(\xi)=\max\{\xi,V_n\}$;

(iv) ΓU is decreasing and concave with $\Gamma U(V_0)>\beta\pi-(1-\beta)C_0$ and $(\Gamma U)'(V_0)=0$.

Proof. Suppose that U is decreasing and concave with $U(V_0)>\beta\pi-(1-\beta)C_0$ and $U'(V_0)=0$.

Step 1 We show $\alpha\geq 0$.

Suppose $\alpha < 0$. Then (57) implies $\mu = (1-\beta) - \alpha u'(c) > 0$, which in turn implies $c = 0$ by (62). Furthermore, given that U is decreasing, for all $\xi \in \Phi^*$ with $f^*(\xi)I(\xi) > 0$, (59) implies $\eta(\xi) > 0$, which in turn implies $V_r(\xi) = \max\{\xi, V_0\}$ by (64). Similarly, $V_n = V_0$.

Hence, (7) implies

$$V = u(0) + \beta(1-\delta)\left[p_w \int_{\Phi^*} \max\{\xi, V_0\} dF^*(\xi) + (1-p_w)V_0\right] = V_0$$

where the second equality follows from (3). Hence, we conclude $\alpha \geq 0$.[44]

Step 2 We show $u'(c) = (1-\beta)/\alpha$.

This follows directly from (57), $\alpha \geq 0$ by Step 1, and the Inada conditions on the utility function u.

Step 3 We show that for all $\xi \in \Phi^*$, $V_r(\xi) = \max\{\xi, V_n\}$ with $U'(V_n) \geq -\alpha$ where the equality holds if $V_n < V_{\max}$.

Suppose $I = 0$. Then the choice of V_n is arbitrary, and the existence of $V_n \in [V_0, V_{\max}]$ with $U'(V_n) \geq -\alpha$ is guaranteed by that U is concave with $U'(V_0) = 0$.

Suppose $I = 1$. If $\eta_n > 0$, then (66) implies $V_n = V_0$, which in turn implies $I(U'(V_n) + \alpha) + \eta_n > 0$ given $U'(V_0) = 0$, $\alpha \geq 0$ by Step 1, and $\eta_n > 0$, which contradicts with (61). Hence, we conclude $\eta_n = 0$, which implies $U'(V_n) = -\alpha$ by (61).

Suppose $\int_{\Phi^*} I(\xi) dF^*(\xi) = 0$. Then the choice of $V_r(\xi)$ is arbitrary for all ξ, and we can simply define $V_r(\xi) = \max\{\xi, V_n\}$ for all ξ.

Suppose $\int_{\Phi^*} I(\xi) dF^*(\xi) > 0$. Then for all $\xi \in \Phi^*$ with $f^*(\xi)I(\xi) > 0$, if $\eta(\xi) = 0$, then (59) implies $U'(V_r(\xi)) = -\alpha$; if $\eta(\xi) > 0$, then (59) and (64) imply $V_r(\xi) = \max\{\xi, V_0\} = \xi$ with $U'(V_r(\xi)) \leq -\alpha$.[45]

Given that U is concave, the result then follows.

Step 4 We show that $I = 1$, and there exists $\bar{\xi} \in [V_0, V_{\max}]$ such that for all $\xi \in \Phi^*$,

$$I(\xi) = \begin{cases} 1 & \text{, if } \xi < \bar{\xi} \\ 0 & \text{, if } \xi > \bar{\xi} \end{cases} ;$$

First, (69) implies

$$\begin{aligned} L_n &= U(V_n) - [\beta\pi - (1-\beta)C_0] + \alpha(V_n - V_0) \\ &\geq U(V_n) - [\beta\pi - (1-\beta)C_0] - U'(V_n)(V_n - V_0) \\ &\geq U(V_0) - [\beta\pi - (1-\beta)C_0] \\ &> 0 \end{aligned}$$

[44]The direct result should be that $\alpha \geq 0$ for all $V \in (V_0, V_{\max})$. However, given that α is a continuous function of V, we conclude that $\alpha \geq 0$ for all $V \in \Phi = [V_0, V_{\max})$.

[45]In equilibrium, no firms would post a contract offering an expected utility lower than V_0. Hence, $\xi \geq V_0$ for all ξ with $f^*(\xi) > 0$.

where the first inequality follows from $V_n \geq V_0$ by (14) and $U'(V_n) \geq -\alpha$ by Step 3, the second inequality follows from that U is concave, and the last inequality follows from $U(V_0) > \beta\pi - (1-\beta)C_0$, which in turn implies $I = 1$ by (12), (60) and (67).[46]

Second, given $V_r(\xi) = \max\{\xi, V_n\}$ for all ξ by Step 3, (68) implies

$$L(\xi) = U(\max\{\xi, V_n\}) - [\beta\pi - (1-\beta)C_0] + \alpha(\max\{\xi, V_n\} - \max\{\xi, V_0\}) \tag{71}$$

which, given that U is decreasing and $\alpha \geq 0$ by Step 1, is decreasing with $L(V_0) = L_n > 0$ as shown above. The result then follows from (9), (58) and (67).

Step 5 We show that ΓU is decreasing and concave with $\Gamma U(V_0) > \beta\pi - (1-\beta)C_0$ and $(\Gamma U)'(V_0) = 0$.

Given what have been shown in Step 1-4, we have

$$L(\overline{\xi})(\overline{\xi} - V_{\max}) = 0,$$

$$(U'(V_n) + \alpha)(V_n - V_{\max}) = 0,$$

$$u\left(u'^{-1}\left(\frac{1-\beta}{\alpha}\right)\right) + \beta(1-\delta)\left[p_w\left(\int_{V_0}^{\overline{\xi}} \max\{\xi, V_n\} dF^*(\xi) + \int_{\overline{\xi}}^{V_{\max}} \xi dF^*(\xi)\right) + (1-p_w)V_n\right] = V.$$

Totally differentiating the equations above gives

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} d\overline{\xi} \\ dV_n \\ d\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dV,$$

where

$$a_{11} = \min\{-\alpha, U'(\overline{\xi})\}(\overline{\xi} - V_{\max}) + L(\overline{\xi}) \geq 0,$$

$$a_{12} = (U'(V_n) + \alpha)(\overline{\xi} - V_{\max}) \leq 0,$$

$$a_{13} = (\max\{\overline{\xi}, V_n\} - \overline{\xi})(\overline{\xi} - V_{\max}) \leq 0,$$

$$a_{22} = U''(V_n)(V_n - V_{\max}) + (U'(V_n) + \alpha) \geq 0,$$

$$a_{23} = V_n - V_{\max} \leq 0,$$

$$a_{31} = \beta(1-\delta)p_w f^*(\overline{\xi})(\max\{\overline{\xi}, V_n\} - \overline{\xi}) \geq 0,$$

$$a_{32} = \beta(1-\delta)[p_w F^*(\min\{\overline{\xi}, V_n\}) + (1-p_w)] \geq 0,$$

$$a_{33} = -(1-\beta)u'(c)/(\alpha^2 u''(c)) \geq 0.$$

[46]Here, $L_n > 0$ implies $\lambda_n(1-2I) < 0$ by (60), which, given $\lambda_n \geq 0$ by (67), in turn implies $1 - 2I < 0$, which, given $I \in [0,1]$ by (12), in turn implies $I = 1$.

Furthermore, by Cramer's rule,

$$\frac{d\bar{\xi}}{dV} = \frac{\|\mathbf{A}_1\|}{\|\mathbf{A}\|} \geq 0, \; \frac{dV_n}{dV} = \frac{\|\mathbf{A}_2\|}{\|\mathbf{A}\|} \geq 0, \text{ and } \frac{d\alpha}{dV} = \frac{\|\mathbf{A}_3\|}{\|\mathbf{A}\|} \geq 0,$$

where

$$\|\mathbf{A}\| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{13}a_{22}a_{31} \geq 0,$$

$$\|\mathbf{A}_1\| = -a_{13}a_{22} \geq 0,$$

$$\|\mathbf{A}_2\| = -a_{11}a_{23} \geq 0,$$

$$\|\mathbf{A}_3\| = a_{11}a_{22} \geq 0.$$

Hence, ΓU is decreasing given $(\Gamma U)'(V) = -\alpha \leq 0$, and concave given $(\Gamma U)''(V) = -d\alpha/dV \leq 0$.

Note that the following static contract delivers expected utility V_0 to the agent: the principal pays the agent zero in the current period, and terminates him at the end of the period regardless of whether the agent receives an outside offer. Hence, (6) implies

$$\Gamma U(V_0) \geq (1-\beta)\theta + \beta\delta[\pi - (1-\beta)C_0] + \beta(1-\delta)[\beta\pi - (1-\beta)C_0] > \beta\pi - (1-\beta)C_0$$

where the second inequality follows from $\beta \in [0, 1)$, $\pi \geq 0$, and $\theta > \beta\pi - (1-\beta)C_0$.[47]

■

Lemma 3. *Let $U_0 = \overline{U} \in B(\Phi)$ and $U_{n+1} = \Gamma U_n$ for $n = 0, 1, 2, \cdots$. Then the sequence of decreasing and concave functions $\{U_n\}_{n=0}^{\infty} \subseteq B(\Phi)$ converges pointwisely and monotonically to the principal's value function U^* as n goes to infinity.*

Proof. Given that the mapping Γ is monotonic and preserves monotonicity and concavity by Lemma 2, it is straightforward to show that $\Gamma^n\overline{U}$ is decreasing and concave with $\theta = \overline{U}(V) \geq \Gamma^n\overline{U}(V) \geq \Gamma^{n+1}\overline{U}(V) \geq U^*(V)$ for all V and $n = 1, 2, \cdots$. The result then follows. ■

Lemma 4. *For all $V \in \Phi$, $V_n = V$.*

Proof. Fix V. The proof takes two steps.

Step 1 We show $U^{*\prime}(V) = U^{*\prime}(V_n) = -\alpha$.

The results follows from (70) and Step 3 of the proof of Lemma 2.

Step 2 We show $V_n = V$.

By Step 1, $U^{*\prime}(V) = U^{*\prime}(V_n^t(V)) = -\alpha$ for $t = 1, 2, \cdots$ where $V_n^t(V)$ denotes the agent's continuation expected utility starting from period $t+1$ conditional on retention without counteroffer in period $\tau = 1, \cdots, t$. It implies that $c(V) = c(V_n^t(V)) \equiv c$ by

[47] If $\theta < \beta\pi - (1-\beta)C_0$, then no firms would enter the labor market in the first place.

Step 2 of the proof of Lemma 2, $\overline{\xi}(V) = \overline{\xi}(V_n^t(V)) \equiv \overline{\xi}$ by (71) and Lemma 2. Hence, for $t = 1, 2, \cdots$, (7) implies

$$u(c)+\beta(1-\delta)\left[p_w\left(\int_{V_0}^{\overline{\xi}} \max\{\xi, V_n^t(V)\}dF^*(\xi) + \int_{\overline{\xi}}^{V_{\max}} \xi dF^*(\xi)\right) + (1-p_w)V_n^t(V)\right] = V_n^{t-1}(V),$$

where $V_n^0(V) = V$. Suppose $V_n(V) \neq V$. Given $\beta(1-\delta) \in [0, 1)$, $\{V_n^{t-1}(V)\}_{t=1}^{\infty}$ is either an increasing sequence converging to $\infty > V_{\max}$, or a decreasing sequence converging to $-\infty < V_0$. Therefore, we conclude $V_n(V) = V$. ■

Lemma 5. *The value function U^* is strictly decreasing.*

Proof. Given that U is decreasing and concave by Lemma 3, U is NOT strictly decreasing only if there exists $\overline{V} > V_0$ such that $U(V) = U(V_0)$ for all $V \in [V_0, \overline{V}]$. Thus, for all $V \in [V_0, \overline{V}]$, $\alpha = -U'(V) = 0$ where the first equality follows from (70), which implies $c(V) = 0$ by (57), which given $V_r(\xi) = \max\{\xi, V\}$ for all ξ by Lemma 2 and 4, in turn implies

$$\overline{V} \leq u(0) + \beta(1-\delta)\left[p_w \int_{\Phi^*} \max\{\xi, \overline{V}\}dF^*(\xi) + (1-p_w)\overline{V}\right]$$

by (7), which in turn implies $\overline{V} \leq V_0$ by (3), which contradicts with $\overline{V} > V_0$. Hence, we conclude that the only contract posted in equilibrium offers expected utility V_0. It is then straightforward to show $V_0 = u(0)/[1-\beta(1-\delta)]$ by (3). ■

B Proof of Proposition 2

Lemma 6. *Suppose $\beta < 1$. Then, for all $V \in \Phi$ with $f^*(V) > 0$, $I(\xi) = 1$ for all ξ with $f^*(\xi) > 0$. That is, in equilibrium, the firm would always retain the worker regardless of his outside offer.*

Proof. Take V with $f^*(V) > 0$ as given. Then, given $V_r(\xi) = \max\{\xi, V\}$ for all ξ by Proposition 1, (68) implies that for all $\xi \in \Phi^*$ with $f^*(\xi) > 0$,

$$\begin{aligned} L(\xi) &= U(\max\{\xi, V\}) - [\beta\pi - (1-\beta)C_0] + \alpha(\max\{\xi, V\} - \max\{\xi, V_0\}) \\ &= U(\max\{\xi, V\}) - [\beta\pi - (1-\beta)C_0] + \alpha(\max\{\xi, V\} - \xi) \\ &= U(\max\{\xi, V\}) - [\beta\pi - (1-\beta)C_0] - U'(V)(\max\{\xi, V\} - \xi) \\ &\geq U(\xi) - [\beta\pi - (1-\beta)C_0] \\ &> 0, \end{aligned}$$

where the second equality follows from $\xi \in \Phi^* = \Phi = [V_0, V_{\max})$ by Lemma 1, the third equality follows from $U'(V) = -\alpha$ by (70), the first inequality follows from that U is concave, and the second inequality follows from $U(\xi) \geq \overline{U}(\xi) = \pi \geq \beta\pi - (1-\beta)C_0$ for all ξ with $f^*(\xi) > 0$ where at least one inequality holds strictly.[48] We therefore conclude $I(\xi) = 1$ by (9), (58) and (67). ■

Given Lemma 6, (17) implies $\gamma(\xi) = u$ for all ξ with $f^*(\xi) > 0$. That is, the acceptance probability is the same for all offers actually posted in equilibrium, which implies that these offers, once accepted, must generate the same expected value for the firms who make them. With this, and given that U is strictly decreasing by Proposition 1, the desired result follows.

C Proof of Proposition 3

Suppose the equilibrium is such that all vacant firms, with θ_l or θ_h, post a contract that offers expected utility $V_{\min}$. That is, $F^*(V_{\min}) = 1$. By (3) then, $V_0 = V_{\min}$.

Step 1 We solve for the optimal contract for an individual firm.

Since the worker's outside value is $V_{\min}$ whether he receives an outside offer or not, the worker never has incentives to quit any ongoing contract. Hence, all expected utility $V \in [V_{\min}, V_{\max})$ can be delivered by a contract that offers a constant wage of $u^{-1}([1-\beta(1-\delta)]V)$. In turn, this implies $\Phi = [V_{\min}, V_{\max})$.

Let $\overline{U}_\theta(\xi)$ denote the (normalized) expected value of a vacant firm with productivity $\theta \in \{\theta_l, \theta_h\}$ which posts a contract offering expected utility $\xi \in \Phi$. Let $U_\theta(V)$ denote the (normalized) expected value of a firm with productivity $\theta \in \{\theta_l, \theta_h\}$ which employs a worker with a contract offering expected utility $V \in \Phi$. Given $k = C_0 = 0$, we have

$$\overline{U}_\theta(\xi) = p_f\gamma(\xi)U_\theta(\xi) + (1 - p_f\gamma(\xi))\beta\overline{U}_\theta(\xi), \tag{72}$$

$$U_\theta(V) = (1-\beta)(\theta - u^{-1}([1-\beta(1-\delta)]V)) + \beta[\delta\overline{U}_\theta(V_{\min}) + (1-\delta)U_\theta(V)], \forall V \in \Phi. \tag{73}$$

Next, since $V_0 = V_{\min}$, only unemployed workers would accept a contract offering expected utility $V_{\min}$, which implies $\gamma(V_{\min}) = u$. Thus (72) and (73) can be rewritten as

$$\overline{U}_\theta(V_{\min}) = p_f u U_\theta(V_{\min}) + (1 - p_f u)\beta\overline{U}_\theta(V_{\min}),$$

$$U_\theta(V_{\min}) = (1-\beta)\theta + \beta[\delta\overline{U}_\theta(V_{\min}) + (1-\delta)U_\theta(V_{\min})].$$

Solving these equations gives

$$\frac{\overline{U}_\theta(V_{\min})}{1-\beta} = \frac{p_f u\theta}{[1-\beta(1-\delta)][1-\beta(1-p_f u)] - \beta\delta p_f u}, \tag{74}$$

[48] Given $\beta < 1$, the incumbent firm strictly prefers retaining the worker than terminating him except when $k = 0$, $C_0 = 0$ and $\pi = 0$. However, it is straightforward to show that there does not exist such an equilibrium. The logic goes as follows: if so, then all firms must earn zero profits by offering the same fixed wage of θ. Thus, a firm can earn strictly positive profits by offering $\theta - \varepsilon$ instead, which is only acceptable to unemployed workers.

which, by (73), implies

$$\frac{U_\theta(V)}{1-\beta} = \frac{[1-\beta(1-p_f u)]\theta}{[1-\beta(1-\delta)][1-\beta(1-p_f u)]-\beta\delta p_f u} - \frac{u^{-1}([1-\beta(1-\delta)]V)}{1-\beta(1-\delta)}, \forall V \in \Phi. \tag{75}$$

With the above value calculations, we then know that the firm, with productivity $\theta \in \{\theta_l, \theta_h\}$ and is currently employing a worker at expected utility $V_{\min}$, would match an outside offer $\xi \in \Phi$ if and only if

$$U_\theta(\xi) \geq \beta \overline{U}_\theta(V_{\min}),$$

which, given (74) and (75), is equivalent to

$$\xi \leq \frac{1}{1-\beta(1-\delta)} u\left(\frac{(1-\beta)[1-\beta(1-\delta)]\theta}{[1-\beta(1-\delta)][1-\beta(1-p_f u)]-\beta\delta p_f u}\right) \equiv \overline{\xi}(\theta). \tag{76}$$

That is, it would match its worker's outside offer up to $\overline{\xi}(\theta)$ in order to retain him. Notice that $\overline{\xi}(\theta)$ is strictly increasing in θ and, in particular, $\overline{\xi}(\theta_l) < \overline{\xi}(\theta_h)$.

Step 2 We calculate the function $\gamma(\cdot)$, taking as given that all (other) vacant firms post a contract offering expected utility $V_{\min}$ as solved in Step 1.

Consider an individual vacant firm in the labor market, with any θ. Suppose it posts a contract offering expected utility $\xi \in [V_{\min}, \overline{\xi}(\theta_l)]$. Then the contract would only be accepted by unemployed workers, implying $\gamma(\xi) = u$. Suppose it posts a contract offering expected utility $\xi \in (\overline{\xi}(\theta_l), \overline{\xi}(\theta_h)]$. Then the contract would be accepted by those who are unemployed and those employed at a firm with the low productivity θ_l. This implies $\gamma(\xi) = u + (1-u)q$.[49] Suppose it posts a contract offering expected utility $\xi \in (\overline{\xi}(\theta_h), V_{\max})$. Then the contract would be accepted by all workers, employed and unemployed, implying $\gamma(\xi) = 1$. To summarize,

$$\gamma(\xi) = \begin{cases} u, & \text{if } \xi \in [V_{\min}, \overline{\xi}(\theta_l)] \\ u + (1-u)q, & \text{if } \xi \in (\overline{\xi}(\theta_l), \overline{\xi}(\theta_h)] \\ 1, & \text{if } \xi \in (\overline{\xi}(\theta_h), V_{\max}) \end{cases}. \tag{77}$$

Step 3 We derive the desired result. Now in order for there to be a stationary equilibrium where all vacant firms offer the same expected utility $V_{\min}$ to the worker it is matched with, it is necessary and sufficient that

$$V_{\min} \in \arg\max_{\xi \in \Phi} \left\{\overline{U}_\theta(\xi)\right\}, \ \theta \in \{\theta_l, \theta_h\} \tag{78}$$

[49]Note that the fraction of employed workers who are with a firm with productivity θ_l is q. When all firms offer the same expected utility $V_{\min}$, they have the same probability $p_f u$ to hire a new worker, and the same probability δ of losing the worker each period after having hired him. Hence, the fraction of employed workers with low productivity is equal to the fraction of firms with low productivity, which is q.

where, by (72),

$$\overline{U}_\theta(\xi) = \frac{p_f\gamma(\xi)}{1-\beta(1-p_f\gamma(\xi))}U_\theta(\xi),$$

with $\gamma(\xi)$ given by (77).

It is straightforward to show that the low productivity firm never has incentives to offer an expected utility higher than $V_{\min}$. For the high productivity firm, to make (78) hold we need only guarantee that it has no incentives to post a contract offering expected utility $\xi \in (\overline{\xi}(\theta_l), \overline{\xi}(\theta_h)]$. Such ξ, once offered, would induce workers who are currently employed at a low productivity firm to take it. That is, (78) holds if and only if for all $\xi \in (\overline{\xi}(\theta_l), \overline{\xi}(\theta_h)]$,

$$\overline{U}_{\theta_h}(V_{\min}) \geq \overline{U}_{\theta_h}(\xi) = \frac{p_f\gamma(\xi)}{1-\beta(1-p_f\gamma(\xi))}U_{\theta_h}(\xi) = \frac{p_f[u+(1-u)q]}{1-\beta\{1-p_f[u+(1-u)q]\}}U_{\theta_h}(\xi),$$

where the first equality follows from (72) and the second equality follows from (77). Finally, given that $U_{\theta_h}(\xi)$ is a decreasing function of ξ by (75), the above condition holds if and only if

$$\overline{U}_{\theta_h}(V_{\min}) \geq \frac{p_f[u+(1-u)q]}{1-\beta\{1-p_f[u+(1-u)q]\}}U_{\theta_h}(\overline{\xi}(\theta_l)),$$

which, given (74), (75) and (76), can be rewritten as

$$p_f u\theta_h \geq \frac{p_f[u+(1-u)q]}{1-\beta\{1-p_f[u+(1-u)q]\}}\{[1-\beta(1-p_f u)]\theta_h - (1-\beta)\theta_l\},$$

which can be rearranged to read as (19) and the proposition is then proved.

D Proof of Proposition 4

Lemma 7. $\Phi = [V_0, V_{\max})$.

Proof. The proof takes two steps.

Step 1 We show that if $V \in \Phi$, then $V \geq V_0$. Suppose $V \in \Phi$. Then there exists a feasible contract $\{c(V'), I(\xi; V'), V_r(\xi; V'), I(V'), V_n(V') : \xi \in \Phi^* \text{ and } V' \in \Phi\}$ satisfying (20)-(28) such that

$$\begin{aligned} V &= u(c(V)) + \beta(1-\delta)p_w \int_{\Phi^*} I(\xi;V)V_r(\xi;V) + (1-I(\xi;V))\max\{\xi, V_0\}dF^*(\xi) \\ &\quad +\beta(1-\delta)(1-p_w)[I(V)V_n(V) + (1-I(V))V_0] \\ &\geq u(0) + \beta(1-\delta)\left[p_w \int_{\Phi^*} \max\{\xi, V_0\}dF^*(\xi) + (1-p_w)V_0\right] \\ &= V_0 \end{aligned}$$

where the first equality follows from (20), the inequality follows from $c(V) \geq 0$ by (24), $V_r(\xi; V) \geq \max\{\xi, V_0\}$ for all $\xi \in \Phi^*$ with $I(\xi; V) = 1$ by (27), and $V_n(V) \geq V_0$ by

(28), and the last equality follows from (3).

Step 2 We show that there exists a feasible contract satisfying (20)-(28) that attains all $V \in [V_0, V_{\max})$. This contract is constructed as follows: For all $\xi \in \Phi^*$ and all $V \in [V_0, V_{\max})$,

$$I(\xi;V) = \begin{cases} 1 & , \text{if } \xi \leq V \\ 0 & , \text{if } \xi > V \end{cases}, \; V_r(\xi;V) = \max\{\xi, V\}, \; I(V) = 1, \text{ and } V_n(V) = V$$

where $c(V) \geq 0$ is set to be such that (20) holds. It is straightforward to show that such $c(V)$ exists and the lemma is then proven. ■

Lemma 8. *For any feasible contract* $\{c(V), I(\xi;V), V_r(\xi;V), I(V), V_n(V) : \xi \in \Phi^*, \; V \in \Phi\}$*, the following holds: For all* $V \in \Phi$*,*

(a) Suppose that there exists $\xi \in \Phi^*$ *such that* $I(\xi;V) = 1$*. Then*

(i) There exists $\overline{V}_r(V) \in \Phi$ *such that*

$$V_r(\xi;V) = \overline{V}_r(V), \; \forall \xi \text{ with } I(\xi;V) = 1; \tag{79}$$

(ii) For all $\xi \in \Phi^*$*,*

$$I(\xi;V) = \begin{cases} 1 & , \text{ if } \xi < \overline{V}_r(V) \\ 0 & , \text{ if } \xi > \overline{V}_r(V) \end{cases};$$

(iii) If $I(V) = 1$*, then* $\overline{V}_r(V) = V_n(V)$*;*

(iv) If $\overline{V}_r(V) > V_0$*, then* $I(V) = 1$*.*

(b) Suppose $I(\xi;V) = 0$ *for all* $\xi \in \Phi^*$*. Then* $I(V) = 1$ *implies* $V_n(V) \leq \max\{\xi, V_0\}$ *for all* $\xi \in \Phi^*$*.*

Proof. Fix $V \in \Phi$.

(a) Suppose that there exists $\xi \in \Phi^*$ such that $I(\xi;V) = 1$.

(i) Suppose there exist $\xi, \xi' \in \Phi^*$ with $I(\xi;V) = I(\xi';V) = 1$ and $V_r(\xi;V) < V_r(\xi';V)$. Then the worker with outside offer ξ strictly prefers reporting ξ' to get expected utility $V_r(\xi';V)$ than reporting ξ truthfully to get expected utility $V_r(\xi;V)$, violating the incentive compatibility. Thus (79) holds for some $\overline{V}_r(V) \in \Phi$.

(ii) We first show $I(\xi;V) = 1$ for all $\xi < \overline{V}_r(V)$. Suppose there exists $\xi < \overline{V}_r(V)$ with $I(\xi;V) = 0$. That is, the worker receiving an outside offer lower than $\overline{V}_r(V)$ is not retained. Then the worker with outside offer ξ strictly prefers reporting some ξ' with $I(\xi';V) = 1$ to get $V_r(\xi';V) = \overline{V}_r(V)$ than reporting ξ truthfully to get $\max\{\xi, V_0\}$, violating the incentive compatibility constraint.

We then show $I(\xi;V) = 0$ for all $\xi > \overline{V}_r(V)$. Suppose there exists $\xi > \overline{V}_r(V)$ with $I(\xi;V) = 1$. Then the expected utility of the worker with outside offer ξ is

$V_r(\xi;V) = \overline{V}_r(V) < \max\{\xi, V_0\}$ if he reports ξ truthfully where the equality follows from (i), violating the self-enforcing constraint (27).

(iii) Suppose $I(V) = 1$. Suppose $V_n(V) > \overline{V}_r(V)$. Then the worker with outside offer ξ with $I(\xi;V) = 1$ strictly prefers reporting not receiving any outside offer to get $V_n(V)$ than reporting ξ truthfully to get $V_r(\xi;V) = \overline{V}_r(V)$, violating the incentive compatibility constraint. Suppose $V_n(V) < \overline{V}_r(V)$. Then the worker with no outside offer strictly prefers reporting some ξ' with $I(\xi';V) = 1$ to get $V_r(\xi';V) = \overline{V}_r(V)$ than reporting not receiving any outside offer truthfully to get $V_n(V)$, violating the incentive compatibility constraint. To summarize, it must hold that $\overline{V}_r(V) = V_n(V)$.

(iv) Suppose $\overline{V}_r(V) > V_0$ and $I(V) = 0$. Then the worker who has not received any outside offer strictly prefers reporting some ξ' with $I(\xi';V) = 1$ to get $V_r(\xi';V) = \overline{V}_r(V)$ than reporting not receiving any outside offer truthfully to get V_0, violating the incentive compatibility constraint.

(b) Suppose $I(\xi;V) = 0$ for all $\xi \in \Phi^*$. Suppose $I(V) = 1$. Suppose there exists ξ with $V_n(V) > \max\{\xi, V_0\}$. Then the worker with outside offer ξ strictly prefers reporting not receiving any outside offer to get $V_n(V)$ than reporting ξ truthfully to get $\max\{\xi, V_0\}$, violating the incentive compatibility constraint. ■

Lemma 9. *The following holds in equilibrium:*

(i) For all $V \in (V_0, V_{\max})$, $\int_{\Phi^*} I(\xi;V)dF^*(\xi) > 0$, $I(V) = 1$, and $V_n(V) > V_0$;

(ii) $I(V_0) = 1$ and $V_n(V_0) = V_0$.

Proof. Take as given a feasible contract $\{c(V), I(\xi;V), V_r(\xi;V), I(V), V_n(V) : \xi \in \Phi^* \text{ and } V \in \Phi\}$ satisfying (20)-(28).

(i) Let $V \in (V_0, V_{\max})$.

We first show that $\int_{\Phi^*} I(\xi;V)dF^*(\xi) > 0$. Suppose otherwise. Then there are two cases: (i) $I(V) = 0$, which implies $V = V_0$ by (20); (ii) $I(V) = 1$, which implies $V_n(V) \leq \max\{\xi, V_0\}$ for all $\xi \in \Phi^*$ by (b) of Lemma 8, which, given $\Phi^* = \Phi = [V_0, V_{\max})$ by Lemma 7 and $V_n(V) \geq V_0$ by (28), in turn implies $V_n(V) = V_0$, which in turn implies $V = V_0$ by (20).

We next show $I(V) = 1$. Suppose $I(V) = 0$. Then given that there exists ξ such that $I(\xi;V) = 1$ as shown above, $\overline{V}_r(V) = V_0$ by (iv) of (a) of Lemma 8, which implies $V = V_0$ by (20).

We now show $V_n(V) > V_0$. Suppose $V_n(V) = V_0$. Then given that there exists ξ such that $I(\xi;V) = 1$ as shown above, $\overline{V}_r(V) = V_n(V) = V_0$ by (iii) of (a) of Lemma 8, which implies $V = V_0$ by (20).

(ii) Let $V = V_0$.

After the worker reports not receiving any outside offer, the incumbent firm either terminates the worker ($I(V_0) = 0$) or retain the worker ($I(V_0) = 1$) with expected utility $V_n(V_0) = V_0$. Note that the worker is indifferent between being terminated and being retained. The expected profit for the incumbent firm is $\beta\pi - (1-\beta)C_0$ in the case of termination and $U(V_0)$ in the case of retention. Thus, it suffices to show $U(V_0) > \beta\pi - (1-\beta)C_0$. Construct a feasible contract at $V = V_0$ as follows: $c(V_0) = 0$, $I(\xi; V) = 0$ for all ξ, $I(V_0) = 0$, and $V_n(V_0) = V_0$ such that

$$U(V_0) \geq (1-\beta)\theta + \beta(1-\delta)[\beta\pi - (1-\beta)C_0] + \beta\delta[\pi - (1-\beta)C_0] > \beta\pi - (1-\beta)C_0$$

given $\theta \geq \pi \geq \beta\pi - (1-\beta)C_0$ in which at least one inequality holds strictly.[50] ∎

Lemma 10. *The following holds for the optimal contract: For all V,*

(i) $c(V_n(V)) \geq c(V)$;

(ii) $V_n(V) \geq V$ where the strict inequality holds if $f^*(V) > 0$.

Proof. Given (i) and (ii) of the proposition which were already proven, the firm's problem of optimal contracting can be rewritten as: For all $V \in [V_0, V_{\max})$,

$$\begin{aligned} U(V) \;=\; & \max_{c, V_n} (1-\beta)(\theta - c) + \beta\delta\left[\pi - (1-\beta)C_0\right] \\ & +\beta(1-\delta)p_w\{F^*(V_n)U(V_n) + (1 - F^*(V_n))\left[\beta\pi - (1-\beta)C_0\right]\} \\ & +\beta(1-\delta)(1-p_w)U(V_n) \end{aligned}$$

subject to

$$u(c) + \beta(1-\delta)p_w\left(F^*(V_n)V_n + \int_{V_n}^{V_{\max}} \xi dF^*(\xi)\right) + \beta(1-\delta)(1-p_w)V_n = V, \tag{80}$$

$$c \geq 0, \tag{81}$$

$$V_n \geq V_0, \tag{82}$$

where $U(\cdot)$ is the firm's value function.

Let α, μ and $\beta(1-\delta)\gamma$ be the Lagrangian multipliers for the constraints (80)-(82) respectively. Then the Kuhn-Tucker conditions for the above Bellman equation are as follows:

$$-(1-\beta) + \alpha u'(c) + \mu = 0, \tag{83}$$

$$p_w f^*(V_n)\{U(V_n) - [\beta\pi - (1-\beta)C_0]\} + [p_w F^*(V_n) + (1-p_w)](U'(V_n) + \alpha) + \gamma = 0, \tag{84}$$

$$\mu c = 0, \tag{85}$$

$$\gamma(V_n - V_0) = 0, \tag{86}$$

$$\mu, \gamma \geq 0. \tag{87}$$

[50] See footnote 48.

In addition, the Envelope Theorem gives

$$U'(V) = -\alpha. \tag{88}$$

Suppose $c(V) > c(V_n(V)) \geq 0$. Then,

$$\begin{aligned} U'(V_n(V)) + \alpha(V) &= -\alpha(V_n(V)) + \alpha(V) \\ &= -\frac{(1-\beta) - \mu(V_n(V))}{u'(c(V_n(V)))} + \frac{(1-\beta) - \mu(V)}{u'(c(V))} \\ &= -\frac{(1-\beta) - \mu(V_n(V))}{u'(c(V_n(V)))} + \frac{1-\beta}{u'(c(V))} \\ &> 0 \end{aligned}$$

where the first equality follows from (88), the second from (83), the third from $c(V) > 0$ which implies $\mu(V) = 0$ by (85), and the inequality follows from $c(V) > c(V_n(V))$ and $\mu(V_n(V)) \geq 0$ by (87). Hence, (84) implies

$$p_w f^*(V_n(V))\{U(V_n(V)) - [\beta\pi - (1-\beta)C_0]\} + \gamma(V) < 0,$$

which, given $\gamma(V) \geq 0$ by (87), in turn implies

$$f^*(V_n(V)) > 0 \text{ and } U(V_n(V)) - [\beta\pi - (1-\beta)C_0] < 0.$$

That is, there are some vacant firms which post a contract offering expected utility $V_n(V)$ with

$$\overline{U}(V_n(V)) \leq U(V_n(V)) < \beta\pi - (1-\beta)C_0 \leq \pi,$$

where the first inequality follows from (5), which contradicts with the equilibrium definition. Therefore, we conclude that $c(V_n(V)) \geq c(V)$ for all V, which implies $V_n(V) \geq V$ for all V by (80).[51]

Take $V \in [V_0, V_{\max})$ with $f^*(V) > 0$ as given. Suppose $V_n(V) = V$. Then,

$$\begin{aligned} & p_w f^*(V_n(V))\{U(V_n(V)) - [\beta\pi - (1-\beta)C_0]\} \\ & +[p_w F^*(V_n(V)) + (1-p_w)](U'(V_n(V)) + \alpha(V)) + \gamma(V) \\ = \ & p_w f^*(V)\{U(V) - [\beta\pi - (1-\beta)C_0]\} + [p_w F^*(V) + (1-p_w)](U'(V) + \alpha(V)) + \gamma(V) \\ = \ & p_w f^*(V)\{U(V) - [\beta\pi - (1-\beta)C_0]\} + \gamma(V) \\ > \ & 0, \end{aligned}$$

where the second equality follows from (88), and the inequality follows from $f^*(V) > 0$

[51]This result follows from a similar argument used in the proof of Lemma 4.

which implies $U(V) \geq \overline{U}(V) = \pi \geq \beta\pi - (1-\beta)C_0$ in which at least one inequality holds strictly, and $\gamma \geq 0$ by (87). This contradicts with (84). Hence, we conclude $V_n(V) > V$.

E Proof of Proposition 5

Given Proposition 4, in equilibrium for all $V \in \Phi^*$ with $f^*(V) > 0$, the expected value of a firm employing a worker with a contract that promises expected utility V is given by

$$\begin{aligned}
U(V) &= (1-\beta)(\theta - c(V)) + \beta\delta[\pi - (1-\beta)C_0] \\
&\quad +\beta(1-\delta)p_w\{F^*(V_n(V))U(V_n(V)) + (1 - F^*(V_n(V)))[\beta\pi - (1-\beta)C_0]\} \\
&\quad +\beta(1-\delta)(1-p_w)U(V_n(V)) \\
&= (1-\beta)(\theta - c(V)) + \beta(1-\delta)[(1-p_w) + p_w F^*(V_n(V))]U(V_n(V)) \\
&\quad +\beta[\delta + \beta(1-\delta)p_w(1 - F^*(V_n(V)))]\pi - \beta(1-\beta)[\delta + (1-\delta)p_w(1 - F^*(V_n(V)))]C_0.
\end{aligned}$$

Next, for all $V \in \Phi^*$ with $f^*(V) > 0$, the expected value of a vacant firm posting a contract that offers expected utility V is

$$\overline{U}(V) = \frac{-(1-\beta)k + p_f\gamma(V)U(V)}{1 - (1 - p_f\gamma(V))\beta}.$$

Given the above, we now derive the conditions that characterize a stationary equilibrium. First, the zero-profit condition (i.e., $\pi = \overline{U}(V) = 0 \ \forall V \in \Phi^*$ with $f^*(V) > 0$) is written as

$$U(V) = \frac{(1-\beta)k}{p_f\gamma(V)}, \ \forall V \in \Phi^* \text{ with } f^*(V) > 0, \tag{89}$$

or, equivalently,

$$\begin{aligned}
\theta - c(V) &= \beta[\delta + (1-\delta)p_w(1 - F^*(V_n(V)))]C_0 \\
&\quad + \frac{1}{p_f\gamma(V)}\left\{1 - \frac{\beta(1-\delta)[(1-p_w) + p_w F^*(V_n(V))]\gamma(V)}{\gamma(V_n(V))}\right\}k, \\
&\quad \forall V \in \Phi^* \text{ with } f^*(V) > 0.
\end{aligned} \tag{90}$$

Second, the stationarity of the distribution of employed workers, G, requires

$$(1-u)G(V_n(V)) = (1-\delta)[(1-p_w) + p_w F^*(V)](1-u)G(V) + (1-\delta)p_w F^*(V)u, \tag{91}$$

where the LHS is the flow out of employment, the RHS flow into employment.

Third, the optimality of the equilibrium contract is given in the following first order condition for the employed worker's next period expected utility $V_n(V)$:

$$\begin{aligned} & p_w f^*(V_n(V))\{U(V_n(V)) - [\beta\pi - (1-\beta)C_0]\} \\ = \quad & (1-\beta)[(1-p_w) + p_w F^*(V_n(V))]\left(\frac{1}{u'(c(V_n(V)))} - \frac{1}{u'(c(V))}\right), \ \forall V \in \Phi^*. \end{aligned} \tag{92}$$

In (92), $U(V_n(V)) - [\beta\pi - (1-\beta)C_0]$ measures the firm's gains from retaining the worker with a continuation contract offering expected utility $V_n(V)$, instead of terminating him. The gains are strictly positive given that the firm would have to go through a costly process after termination to find a new worker identical to the departing worker. Notice next that the absolute value of the term

$$\frac{1}{u'(c(V_n(V)))} - \frac{1}{u'(c(V))}$$

measures the firm's costs, in units of current period consumption, of deviating from perfectly smoothing the worker's consumptions across the current and the next periods, conditional on retaining the worker. Notice that this term is zero if and only $V_n(V) = V$ so the worker's consumption is constant between the current and the next period. Thus (92) simply equates the expected gains and costs associated with an increase in $V_n(V)$ at the margin.

To summarize, a stationary equilibrium of the model for the case of $\beta < 1$ and private outside offers is characterized by a tuple

$$\{u, p_w, p_f, [U(V), V_n(V), F^*(V), G(V) : V \in \Phi^*]\}$$

that solves equations (1), (2), (17), and (89)-(92).

Suppose that the worker would reject any outside offer that offers the same expected utility as the ongoing contract.

(i) Suppose that the distribution F^* has a mass at $\overline{V}$ (hence, the distribution G must also have a mass, say $\kappa > 0$, at $\overline{V}$).

It is straightforward to show that the optimal contract offering expected utility $\overline{V}$ is a fixed wage contract such that $V_n(\overline{V}) = \overline{V}$. Hence, $\gamma(\overline{V}) = u + (1-u)(1-\kappa)$ and

$$U(\overline{V}) = \frac{(1-\beta)(\theta - c(\overline{V})) + \beta\delta[\pi - (1-\beta)C_0]}{1-\beta(1-\delta)}. \tag{93}$$

Consider a fixed wage contract paying the worker $c(\overline{V}) + \varepsilon$ for some $\varepsilon > 0$, which offers an expected utility strictly greater than $\overline{V}$. The expected value of a firm which employs a worker with this newly constructed fixed wage contract is

$$\widehat{U} \;=\; \frac{(1-\beta)[\theta-(c(\overline{V})+\varepsilon)]+\beta\delta[\pi-(1-\beta)C_0]}{1-\beta(1-\delta)} = U(\overline{V}) - \frac{(1-\beta)\varepsilon}{1-\beta(1-\delta)} \quad (94)$$

where the second equality follows from (93). The probability of acceptance for this newly constructed fixed wage contract is 1. Hence, the expected value of a firm which posts this newly constructed fixed wage contract is

$$\frac{-(1-\beta)k+p_f\widehat{U}}{1-(1-p_f)\beta},$$

while the expected value of a firm which posts the contract offering expected utility $\overline{V}$ is

$$\overline{U}(\overline{V}) = \frac{-(1-\beta)k+p_f\gamma(\overline{V})U(\overline{V})}{1-(1-p_f\gamma(\overline{V}))\beta}.$$

Hence,

$$\begin{aligned} & \frac{-(1-\beta)k+p_f\widehat{U}}{1-(1-p_f)\beta} - \frac{-(1-\beta)k+p_f\gamma(\overline{V})U(\overline{V})}{1-(1-p_f\gamma(\overline{V}))\beta} \\ = \; & \frac{\beta p_f(1-\gamma(\overline{V}))(1-\beta)k+p_f(1-\beta)(1-\gamma(\overline{V}))U(\overline{V})}{[1-(1-p_f)\beta][1-(1-p_f\gamma(\overline{V}))\beta]} - \frac{p_f(1-\beta)\varepsilon}{[1-\beta(1-\delta)][1-(1-p_f)\beta]} \end{aligned}$$

given (94). Furthermore, given $k>0$, $\gamma(\overline{V}) = u+(1-u)(1-\kappa) < 1$ due to $\kappa > 0$, and $U(\overline{V}) \geq 0$, it is straightforward to show that there exists $\varepsilon > 0$ such that it is optimal for vacant firms to post the newly constructed fixed wage contract instead of the contract offering expected utility $\overline{V}$. This is a contradiction.

(ii) The proof takes two steps.

Step 1 We show $V_n(\underline{V}) = \underline{V}$.

Suppose $V_n(\underline{V}) > \underline{V}$. Then, it is straightforward to show that for all $V \in [\underline{V}, V_n(\underline{V})]$, $G(V) = 0$ by (91).

Hence, for all $V \in [\underline{V}, V_n(\underline{V})]$, (17) implies

$$\gamma(V) = u+(1-u)G(V) = u,$$

which, given $\pi = 0$, in turn implies

$$U(V) = \frac{(1-\beta)k}{p_f\gamma(V)} = \frac{(1-\beta)k}{p_f u}$$

by (89), which in turn implies $\alpha(V) = -U'(V) = 0$ by (70), which in turn implies $c(V) = 0$ by (83).

Given $G(V) = 0$ for all $V \in [\underline{V}, V_n(\underline{V})]$ as shown above, (91) implies that for all $V \in [\underline{V}, V_n(\underline{V})]$,

$$(1-u)G(V_n(V)) = (1-\delta)p_w F^*(V)u,$$

which in turn implies

$$G(V_n(V)) = \frac{(1-\delta)p_w u}{1-u}F^*(V). \tag{95}$$

Given $\pi = 0$ and $\alpha(V) = -U'(V) = 0$ for all $V \in [\underline{V}, V_n(\underline{V})]$ by (70), the first order condition (84) implies that for all $V \in [\underline{V}, V_n(\underline{V})]$,

$$p_w f^*(V_n(V))[U(V_n(V)) + (1-\beta)C_0] + [(1-p_w) + p_w F^*(V_n(V))]U'(V_n(V)) = 0^{52},$$

which in turn implies

$$\frac{p_w f^*(V_n(V))}{(1-p_w) + p_w F^*(V_n(V))} = -\frac{U'(V_n(V))}{U(V_n(V)) + (1-\beta)C_0}.$$

Hence, there exists a constant X such that for all $V \in [\underline{V}, V_n(\underline{V})]$,

$$\ln[(1-p_w) + p_w F^*(V_n(V))] = \ln\left[\frac{X}{U(V_n(V)) + (1-\beta)C_0}\right],$$

which implies

$$\begin{aligned}
F^*(V_n(V)) &= \frac{1}{p_w}\left[\frac{X}{U(V_n(V)) + (1-\beta)C_0} - (1-p_w)\right] \\
&= \frac{1}{p_w}\left[\frac{X}{\frac{(1-\beta)k}{p_f\gamma(V_n(V))} + (1-\beta)C_0} - (1-p_w)\right] \\
&= \frac{1}{p_w}\left\{\frac{X}{\frac{(1-\beta)k}{p_f u[1+(1-\delta)p_w F^*(V)]} + (1-\beta)C_0} - (1-p_w)\right\}
\end{aligned} \tag{96}$$

where the second equality follows from $U(V_n(V)) = (1-\beta)k/(p_f\gamma(V_n(V)))$ by (89), and the third equality follows from $\gamma(V_n(V)) = u + (1-u)G(V_n(V)) = u[1+(1-\delta)p_w F^*(V)]$ by (17) and (95).

Finally, inserting $c(V) = 0$, $\gamma(V) = u$, (95) and (96) into (90) implies that for all $V \in [\underline{V}, V_n(\underline{V})]$, $F^*(V)$ is a constant, which implies $f^*(V) = 0$. This is a contradiction with the definition of $\underline{V}$. Hence, we conclude $V_n(\underline{V}) = \underline{V}$.

Step 2 We show $f^*(\underline{V}) = 0$.

Given $V_n(\underline{V}) = \underline{V}$ by Step 1, (84) implies

$$p_w f^*(\underline{V})\{U(\underline{V}) - [\beta\pi - (1-\beta)C_0]\} + [(1-p_w) + p_w F^*(\underline{V})](U'(\underline{V}) + \alpha) = 0,$$

which, given $\pi = 0$, $U'(\underline{V}) = -\alpha$ by (70), and $U(\underline{V}) - [\beta\pi - (1-\beta)C_0] = (1-\beta)k/(p_f\gamma(\underline{V})) + (1-\beta)C_0 > 0$ by (89), in turn implies $f^*(\underline{V}) = 0$.

[52] Given $V_n(V) \geq V \geq V_0$ by (ii) of Lemma 10, the feasibility constraint (82) is not binding such that $\gamma = 0$.

It is straightforward to show $V_n(\overline{V}) = \overline{V}$ since the optimal contract offering the highest expected utility is a fixed wage contract. Hence, it can be shown $f^*(\overline{V}) = 0$.

F Proof of Proposition 6

(i) Suppose $k = C_0 = 0$. Then (90) implies that $c(V) = \theta$ for all V. That is, all firms offer the same Walrasian wage $w = \theta$.

(ii) Given $V_n(\underline{V}) = \underline{V}$, $V_n(\overline{V}) = \overline{V}$ and $c(\underline{V}) = 0$ as shown in Proof of Proposition 5, (90) implies

$$\theta = \beta[\delta + (1-\delta)p_w]C_0 + \frac{1-\beta(1-\delta)(1-p_w)}{p_f u}k,$$

$$\theta - c(\overline{V}) = \beta\delta C_0 + \frac{1-\beta(1-\delta)}{p_f}k.$$

Furthermore, given $V_n(\overline{V}) = \overline{V}$, (91) implies

$$u = \frac{\delta}{\delta + (1-\delta)p_w},$$

which in turn implies that p_w and u move on opposite directions. Given that p_w move on opposite directions with both u and p_f, it is then straightforward to show that $c(\overline{V})$ is a decreasing function of k and C_0.

G Proof of Proposition 7

Lemma 11. *Suppose $\beta = 1$. Then the following holds in the optimal contract: for all $V \in \Phi^*$ with $f^*(V) > 0$,*

(i) $V_r(\xi) = \max\{\xi, V\}$ for all $V \in \Phi^*$ and $V_n = V$;

(ii) $I = 1$, and for all $\xi \in \Phi^*$,

$$I(\xi) = \begin{cases} 1 & , \text{ if } \xi < V \\ \text{any } 0 \text{ or } 1 & , \text{ if } \xi > V \end{cases}.$$

Proof. (i) the result follows from $V_r = \max\{\xi, V_n\}$ for all $\xi \in \Phi^*$ by (iii) of Lemma 2 and $V_n = V$ by Lemma 4 directly.

(ii) Fix $V \in \Phi^*$ with $f^*(V) > 0$. Note that $f^*(V) > 0$ implies that it is optimal for vacant firms to post a contract offering expected utility V in equilibrium.

First, $I = 1$ follows from (ii) of Lemma 2.

Second, (71) implies that for all $\xi \in \Phi^*$ with $f^*(\xi) > 0$,

$$\begin{aligned} L(\xi) &= U(\max\{\xi, V\}) - [\beta\pi - (1-\beta)C_0] + \alpha(\max\{\xi, V\} - \max\{\xi, V_0\}) \\ &= U(\max\{\xi, V\}) - [\beta\pi - (1-\beta)C_0] + \alpha(\max\{\xi, V\} - \xi) \end{aligned}$$

where the second equality follows from $\xi \in \Phi^* = [V_0, V_{\max})$ by Lemma 1.

Suppose $\xi < V$. Then

$$L(\xi) = U(V) - [\beta\pi - (1-\beta)C_0] + \alpha(V - \xi) \geq 0.$$

That is, the principal weakly prefers retention than termination. However, the agent strictly prefers retention (with continuation expected utility V) than termination (with expected utility ξ). Hence, $I(\xi) = 1$.

Suppose $\xi > V$. Then

$$\begin{aligned} L(\xi) &= U(\xi) - [\beta\pi - (1-\beta)C_0] \\ &= \frac{(1-\beta)k + [1-(1-p_f\gamma(\xi))\beta]\overline{U}(\xi)}{p_f\gamma(\xi)} - [\beta\pi - (1-\beta)C_0] \\ &= \overline{U}(\xi) - \pi \\ &= 0, \end{aligned}$$

where the second equality follows from the definition of $\overline{U}(\xi)$, the third equality follows from $\beta = 1$, and the last equality follows from $f^*(\xi) > 0$ which implies that the expected value of vacant firms posting a contract offering expected utility ξ is π. That is, the principal is indifferent between retention (by matching his outside offer) and termination, so is the agent (either way, the agent's expected utility is ξ). ■

H Proof of Proposition 8

Suppose $0 < C_0 + k < \theta/\delta$.

(i) From equation (42), it is straightforward to show

$$\widetilde{f}^*(c) = \frac{1}{(1-\delta)p_w}\left[C_0^2 + \frac{4k(\theta - c)}{p_f\delta}\right]^{-\frac{1}{2}} > 0,$$

$$\widetilde{f}^{*\prime}(c) = \frac{2k}{(1-\delta)p_w p_f\delta}\left[C_0^2 + \frac{4k(\theta - c)}{p_f\delta}\right]^{-\frac{3}{2}} > 0,$$

which in turn imply that the density function $\widetilde{f}^*(\cdot)$ is increasing and convex.

(ii) It can be derived from (42) and (44) that

$$\widetilde{f}^*(c) = \frac{1}{(1-\delta)p_w}\left[C_0^2 + \frac{4k(\theta - c)}{p_f\delta}\right]^{-\frac{1}{2}}$$

$$d\widetilde{f}^*(c) = \underbrace{-\frac{1}{(1-\delta)p_w}\left[C_0^2 + \frac{4k(\theta - c)}{p_f\delta}\right]^{-\frac{3}{2}}}_{(-)} \underbrace{\left\{\frac{1}{p_w}\left[C_0^2 + \frac{4k(\theta - c)}{p_f\delta}\right]dp_w + C_0 dC_0 + \frac{2(\theta - c)}{\delta}d(k/p_f)\right\}}. \tag{97}$$

Case 1: $dC_0 > 0$ and $dk = 0$.

Given (43), (44) and the fact that p_w and p_f move in opposite directions, it is straightforward to show $dp_w < 0$, $dp_f > 0$, $d(k/p_f) < 0$ and $d\bar{c} < 0$. Given $dp_w < 0$ and $d(k/p_f) < 0$, the second part in (97) is increasing in c. Furthermore, given that the first part in (97) is negative, the following holds:

(a) Suppose $d\widetilde{f}^*(\bar{c}) \geq 0$. Then how an increase in C_0 changes the density of the equilibrium wages offered is as illustrated in Figure 2(a);

(b) Suppose $d\widetilde{f}^*(\bar{c}) < 0$. Then how an increase in C_0 changes the density of the equilibrium wages offered is as illustrated in Figure 2(b).[53]

Case 2: $dk > 0$ and $dC_0 = 0$.

Given (43), (44) and the fact that p_w and p_f move in opposite directions, it is straightforward to show $dp_w < 0$, $dp_f > 0$, $d(k/p_f) > 0$ and $d\bar{c} < 0$. Given $dC_0 = 0$, rewrite (97) as

$$d\widetilde{f}^*(c) = \underbrace{-\frac{2(\theta - c)}{(1-\delta)\delta p_w}\left[C_0^2 + \frac{4k(\theta - c)}{p_f\delta}\right]^{-\frac{3}{2}}}_{(-)}\underbrace{\left\{\left[\frac{\delta C_0^2}{2p_w(\theta - c)} + \frac{2k}{p_w p_f}\right]dp_w + d(k/p_f)\right\}}. \tag{98}$$

Thus, given $dp_w < 0$, the second part in (98) is decreasing in c. Moreover, given that the first part in (98) is negative, it can be shown that

(a) if $d\widetilde{f}^*(\underline{c}) \geq 0$, then how an increase in k changes the density of the equilibrium wages offered is as illustrated in Figure 2(c); and

(b) if $d\widetilde{f}^*(\underline{c}) < 0$, then how an increase in k changes the density of the equilibrium wages offered is as illustrated in Figure 2(d).

Case 3: $dC_0 > 0$, $k = 0$ and $dk = 0$.

Given (43), (44) and the fact that p_w and p_f move in opposite directions, it is straightforward to show $dp_w < 0$, $dp_f > 0$, $d(k/p_f) = 0$ and $d\bar{c} < 0$. Given $k = 0$ and $d(k/p_f) = 0$, rewrite (97) as

$$d\widetilde{f}^*(c) = -\frac{C_0^{-2}}{(1-\delta)p_w^2}(C_0 dp_w + p_w dC_0) > 0.$$ [54]

Note that $d\widetilde{f}^*(c)$ is independent of c.

(iii) Suppose $k = 0$. Then (41) implies

$$\widetilde{F}^*(c) = 1 - \frac{1}{(1-\delta)p_w}\left(\frac{\theta - c}{C_0} - \delta\right).$$

[53] Given $d\bar{c} < 0$, it is straightforward to rule out the possibility of $d\widetilde{f}^*(c) < 0$ for all c, and so the density after the increase in C_0 is below the initial density uniformly.

[54] Given $k = 0$, (44) implies $\theta = [\delta + (1-\delta)p_w]C_0$, which, given $dC_0 > 0$, in turn implies $C_0 dp_w + p_w dC_0 = d(p_w C_0) < 0$.

In addition, given $\underline{c} = 0$, (41) implies

$$p_w = \frac{1}{1-\delta}\left(\frac{\theta}{C_0} - \delta\right) \in [0,1],$$

and the desired result then follows.

I Equations used in the calibration

This appendix shows how to solve for the values of $\underline{V}$, $\overline{V}$ and $c(\underline{V})$, $c(\overline{V})$, and the functions $V_n(\cdot)$, $F^*(\cdot)$, $G(\cdot)$, $U(\cdot)$, $\gamma(\cdot)$, and $c(\cdot)$.

To prepare for the derivations that follow, remember the expected utility of unemployed workers is defined as

$$V_0 = u(b) + \beta(1-\delta)\left[p_w^u \int_{\underline{V}}^{\overline{V}} \max\{\xi, V_0\} dF^*(\xi) + (1-p_w^u)V_0\right]. \tag{99}$$

The expected value of a firm employing a worker with a wage-tenure contract offering expected utility V is

$$\begin{aligned} U(V) &= (1-\beta)(\theta - c(V)) + \beta[\delta + (1-\delta)\lambda][\pi - (1-\beta)C_0] \\ &\quad + \beta(1-\delta)(1-\lambda)p_w^e\{F^*(V_n(V))U(V_n(V)) + (1-F^*(V_n(V)))[\beta\pi - (1-\beta)C_0]\} \\ &\quad + \beta(1-\delta)(1-\lambda)(1-p_w^e)U(V_n(V)), \end{aligned} \tag{100}$$

where

$$\begin{aligned} V &= u((1-\tau)c(V)) + \beta(1-\delta)(1-\lambda)\left[p_w^e \int_{\underline{V}}^{\overline{V}} \max\{\xi, V_n(V)\} dF^*(\xi) + (1-p_w^e)V_n(V)\right] \\ &\quad + \beta(1-\delta)\lambda\left[p_w^u \int_{\underline{V}}^{\overline{V}} \max\{\xi, V_0\} dF^*(\xi) + (1-p_w^u)V_0\right]. \end{aligned} \tag{101}$$

And last, the stationarity condition for an equilibrium is

$$(1-u)G(V_n(V)) = (1-\delta)(1-\lambda)[(1-p_w^e) + p_w^e F^*(V)](1-u)G(V) + (1-\delta)(1-\lambda)p_w^u F^*(V)u. \tag{102}$$

We now solve for the values of $\underline{V}$, $\overline{V}$, $c(\underline{V})$, and $c(\overline{V})$. Given $\underline{V} = V_0$ and $V_n(\underline{V}) = \underline{V}$ as shown in the proof of Proposition 5, we have, from (99),

$$\int_{\underline{V}}^{\overline{V}} \max\{\xi, \underline{V}\} dF^*(\xi) = \frac{[1-\beta(1-\delta)(1-p_w^u)]\underline{V} - u(b)}{\beta(1-\delta)p_w^u}.$$

Combining this with (101) and letting $V = \underline{V}$ and $\overline{V}$ respectively, we have

$$u((1-\tau)c(\underline{V})) = \frac{(1-\lambda)p_w^e + \lambda p_w^u}{p_w^u} u(b) + \frac{(1-\lambda)[1-\beta(1-\delta)](p_w^u - p_w^e)}{p_w^u}\underline{V}, \tag{103}$$

$$u((1-\tau)c(\overline{V})) = [1-\beta(1-\delta)(1-\lambda)]\overline{V} - \lambda(\underline{V} - u(b)). \tag{104}$$

Next, inserting the free entry and exit condition

$$\pi = \frac{-(1-\beta)k + p_f\gamma(V)U(V)}{1-(1-p_f\gamma(V))\beta} = 0$$

into (100) gives

$$\begin{aligned} \theta - c(V) &= \beta[\delta + (1-\delta)\lambda + (1-\delta)(1-\lambda)p_w^e(1-F^*(V_n(V)))]C_0 \\ &+ \frac{1}{p_f\gamma(V)}\left\{1 - \frac{\beta(1-\delta)(1-\lambda)[(1-p_w^e)+p_w^eF^*(V_n(V))]\gamma(V)}{\gamma(V_n(V))}\right\}k. \end{aligned}$$

Letting $V = \underline{V}$, $\overline{V}$ respectively, we have

$$c(\underline{V}) = \theta - \beta[\delta + (1-\delta)\lambda + (1-\delta)(1-\lambda)p_w^e]C_0 - \frac{1-\beta(1-\delta)(1-\lambda)(1-p_w^e)}{p_f\gamma(\underline{V})}k, \tag{105}$$

$$c(\overline{V}) = \theta - \beta[\delta + (1-\delta)\lambda]C_0 - \frac{1-\beta(1-\delta)(1-\lambda)}{p_f}k. \tag{106}$$

By now then, $\underline{V}$, $\overline{V}$, $c(\underline{V})$ and $c(\overline{V})$ can be solved from the equations (103)-(106).

Next, we solve a set of differential equations for the six functions $V_n(\cdot)$, $F^*(\cdot)$, $G(\cdot)$, $U(\cdot)$, $\gamma(\cdot)$, and $c(\cdot)$. For all $V \in [\underline{V}, \overline{V}]$, we have

$$\begin{aligned} & p_w^e f^*(V_n(V))\{U(V_n(V)) - [\beta\pi - (1-\beta)C_0]\} \\ = \; & [(1-p_w^e) + p_w^eF^*(V_n(V))]\left(U'(V) - U'(V_n(V))\right), \end{aligned}$$

$$\begin{aligned} \theta - c(V) &= \beta[\delta + (1-\delta)\lambda + (1-\delta)(1-\lambda)p_w^e(1-F^*(V_n(V)))]C_0 \\ &+ \frac{1}{p_f\gamma(V)}\left\{1 - \frac{\beta(1-\delta)(1-\lambda)[(1-p_w^e)+p_w^eF^*(V_n(V))]\gamma(V)}{\gamma(V_n(V))}\right\}k, \end{aligned}$$

and

$$(1-u)G(V_n(V)) = (1-\delta)(1-\lambda)[(1-p_w^e)+p_w^eF^*(V)](1-u)G(V) + (1-\delta)(1-\lambda)p_w^uF^*(V)u,$$

where the first equation is the first order condition for the optimal wage-tenure contract, the second equation is the free entry and exit condition as derived above, the third equation is the stationarity condition (102). The above three equations, together with the promise-keeping constraint (101), plus

$$U(V) = \frac{(1-\beta)k}{p_f\gamma(V)}, \; \forall V \in [\underline{V}, \overline{V}],$$

and

$$\gamma(V) = \frac{p_w^e(1-u)G(V) + p_w^u u}{p_w^e(1-u) + p_w^u u}, \forall V \in [\underline{V}, \overline{V}],$$

a total of six equations, then allow us to solve for simultaneously for the six functions $V_n(\cdot)$, $F^*(\cdot)$, $G(\cdot)$, $U(\cdot)$, $\gamma(\cdot)$, and $c(\cdot)$.

J How does the cost of job turnover affect the equilibrium distribution of wages offered?

In equilibrium, the mean of the wages offered is given by

$$\begin{aligned}
\mathbb{E}(c) &= \int_{\underline{c}}^{\overline{c}} c d\widetilde{F}^*(c) \\
&= \int_0^{\theta-\delta\left(C_0+\frac{k}{p_f}\right)} \frac{c}{(1-\delta)p_w} \left[C_0^2 + \frac{4k(\theta-c)}{p_f\delta}\right]^{-\frac{1}{2}} dc \\
&= -\frac{p_f\delta}{2k(1-\delta)p_w} \int_0^{\theta-\delta\left(C_0+\frac{k}{p_f}\right)} cd\left[C_0^2 + \frac{4k(\theta-c)}{p_f\delta}\right]^{\frac{1}{2}} \\
&= -\frac{p_f\delta}{2k(1-\delta)p_w} \left\{\left[\theta - \delta\left(C_0 + \frac{k}{p_f}\right)\right]\left(C_0 + 2\frac{k}{p_f}\right) - \int_0^{\theta-\delta\left(C_0+\frac{k}{p_f}\right)} \left[C_0^2 + \frac{4k(\theta-c)}{p_f\delta}\right]^{\frac{1}{2}} dc\right\} \\
&= \frac{p_f\delta}{2k(1-\delta)p_w} \left\{\frac{p_f\delta}{6k}\left[\left(C_0^2 + \frac{4k\theta}{p_f\delta}\right)^{\frac{3}{2}} - \left(C_0 + 2\frac{k}{p_f}\right)^3\right] - \left[\theta - \delta\left(C_0 + \frac{k}{p_f}\right)\right]\left(C_0 + 2\frac{k}{p_f}\right)\right\}.
\end{aligned}$$

Suppose $C_0 = 0$. Then, (44) implies

$$\frac{k}{p_f} = \frac{\delta\theta}{[\delta + (1-\delta)p_w]^2},$$

which in turn implies

$$\mathbb{E}(c) = \frac{2\theta}{3} - \frac{\delta\theta[2\delta + (1-\delta)p_w]}{3[\delta + (1-\delta)p_w]^2},$$

which is an increasing function of p_w. Therefore, as k goes up, the probability for a worker to meet a firm p_w goes down (as shown in **Case 2**), and so the mean wage offered, $\mathbb{E}(c)$, goes down as well.

And the variance of the wages offered, $Var(c)$, is

$$
\begin{aligned}
& \int_{\underline{c}}^{\overline{c}} (c - \mathbb{E}(c))^2 d\widetilde{F}^*(c) \\
= \; & \int_{\underline{c}}^{\overline{c}} c^2 d\widetilde{F}^*(c) - (\mathbb{E}(c))^2 \\
= \; & \int_0^{\theta - \delta\left(C_0 + \frac{k}{p_f}\right)} \frac{c^2}{(1-\delta)p_w} \left[C_0^2 + \frac{4k(\theta - c)}{p_f \delta}\right]^{-\frac{1}{2}} dc - (\mathbb{E}(c))^2 \\
= \; & -\frac{p_f \delta}{2k(1-\delta)p_w} \int_0^{\theta - \delta\left(C_0 + \frac{k}{p_f}\right)} c^2 d\left[C_0^2 + \frac{4k(\theta - c)}{p_f \delta}\right]^{\frac{1}{2}} - (\mathbb{E}(c))^2 \\
= \; & -\frac{p_f \delta}{2k(1-\delta)p_w} \left\{ \left[\theta - \delta\left(C_0 + \frac{k}{p_f}\right)\right]^2 \left(C_0 + 2\frac{k}{p_f}\right) - \int_0^{\theta - \delta\left(C_0 + \frac{k}{p_f}\right)} \left[C_0^2 + \frac{4k(\theta - c)}{p_f \delta}\right]^{\frac{1}{2}} dc^2 \right\} \\
& -(\mathbb{E}(c))^2 \\
= \; & -\frac{p_f \delta}{2k(1-\delta)p_w} \left\{ \left[\theta - \delta\left(C_0 + \frac{k}{p_f}\right)\right]^2 \left(C_0 + 2\frac{k}{p_f}\right) + \frac{p_f \delta}{3k} \int_0^{\theta - \delta\left(C_0 + \frac{k}{p_f}\right)} cd\left[C_0^2 + \frac{4k(\theta - c)}{p_f \delta}\right]^{\frac{3}{2}} \right\} \\
& -(\mathbb{E}(c))^2 \\
= \; & -\frac{p_f \delta}{2k(1-\delta)p_w} \left\{ \left[\theta - \delta\left(C_0 + \frac{k}{p_f}\right)\right]^2 \left(C_0 + 2\frac{k}{p_f}\right) + \frac{p_f \delta}{3k} \left[\theta - \delta\left(C_0 + \frac{k}{p_f}\right)\right] \left(C_0 + 2\frac{k}{p_f}\right)^3 \right\} \\
& -\frac{p_f^3 \delta^3}{60k^3(1-\delta)p_w} \int_0^{\theta - \delta\left(C_0 + \frac{k}{p_f}\right)} d\left[C_0^2 + \frac{4k(\theta - c)}{p_f \delta}\right]^{\frac{5}{2}} - (\mathbb{E}(c))^2 \\
= \; & -\frac{p_f \delta}{2k(1-\delta)p_w} \left\{ \left[\theta - \delta\left(C_0 + \frac{k}{p_f}\right)\right]^2 \left(C_0 + 2\frac{k}{p_f}\right) + \frac{p_f \delta}{3k} \left[\theta - \delta\left(C_0 + \frac{k}{p_f}\right)\right] \left(C_0 + 2\frac{k}{p_f}\right)^3 \right\} \\
& +\frac{p_f^3 \delta^3}{60k^3(1-\delta)p_w} \left[\left(C_0^2 + \frac{4k\theta}{p_f \delta}\right)^{\frac{5}{2}} - \left(C_0 + 2\frac{k}{p_f}\right)^5 \right] - (\mathbb{E}(c))^2.
\end{aligned}
$$

It is difficult to show analytically that $Var(c)$ is generally a decreasing function of C_0 and k - the above expression being complicated, but the many numerical examples we computed do suggest this be the case, for all the parameter values that we picked and considered reasonable.

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